Univariate continuous distributions: symmetries and transformations

M.C. Jones

Department of Mathematics & Statistics, The Open University, Walton Hall, Milton Keynes, MK7 6AA, U.K.

ABSTRACT

If the univariate random variable X follows the distribution with distribution function F, then so does $Y = F^{-1}(1 - F(X))$. This known result defines the type of (generalised) symmetry of F, which is here referred to as T-symmetry; for example, ordinary symmetry about θ corresponds to $Y = 2\theta - X$. Some distributions, with density f_S, display a density-level symmetry of the form $f_S(x) = f_S(s(x))$, for some monotone transformation function $s(x) \neq x$; I call this S-symmetry. The main aim of this article is to introduce the S-symmetric dual of any (necessarily T-symmetric) F, and to explore the consequences thereof. Chief amongst these are the connections between the random variables following F and f_S, and relationships between measures of ordinary symmetry based on quantiles and on density values.

Keywords: density-based asymmetry; probability integral transformation; quantile-based skewness; R-symmetry; S-symmetry.

1. Introduction

The usual notion of symmetry of a univariate absolutely continuous distribution is that the random variable $X-\theta$ has the same distribution as $\theta-X$ for some centre of symmetry θ. This will be called "ordinary symmetry" in this article.

Even when not ordinary symmetric, a distribution might have an alternative symmetry such as, for non-negative X, log-symmetry, that is, the ordinary symmetry of the distribution of $\log X$. This is equivalent to saying that X has the same distribution as ϕ^2/X for some ϕ. See Seshadri (1965). In fact, as is known, every distribution has a symmetry of this sort: for each distribution, there is a unique continuous decreasing function t such that X has the same distribution as $t(X)$. Here, I shall call this 'T-symmetry'. See Section 2.1.

Recently, there has been interest in an alternative, density-based, symmetry, the identification of a continuous monotone transformation function $s(x) \neq x$ such that $f(x) = f(s(x))$ for all x in the support of the distribution with density f. Here, I shall call this, as I have elsewhere, 'S-symmetry'. See Section 2.2.

The main contribution of this article is to establish a duality between T- and S-symmetry. Specifically, in Section 3.1, I shall identify, for each distribution with a specific T-symmetry, the density of a corresponding distribution with S-symmetry *where the symmetry functions are the same, $s(x) = t(x)$ for all x*. I shall also show how the random variables associated with these dual distributions can be obtained one from the other. After description of a number of special cases and antecedents in Section 4, the remaining main sections of the paper consider some further aspects of the interaction between T- and S-symmetry (Section 5) and some further aspects of T-symmetry alone (Section 6). There are brief closing remarks in Section 7.

2. Background: T- and S-Symmetries

2.1 T-Symmetry

Let the distribution of interest have invertible distribution function F. To identify t, start with the observation that if U is uniformly distributed on $(0,1)$, then $1-U$, and no other function of U, is also distributed as $U(0,1)$. The probability integral transformation (PIT) then tells us that $X = F^{-1}(U)$ follows the distribution F. But because $1-U$ is also uniform on $(0,1)$, it must be the case that $Y = F^{-1}(1-U)$ follows F also. Thus, both X and $Y = t(X)$, where

$$t(x) = F^{-1}(1-F(x)), \tag{1}$$

have the same distribution, F.

For an alternative derivation, consider determining t such that

$$F(x) = P(X \leq x) = P(t(X) \leq x) = P(X \geq t^{-1}(x)) = 1 - F(t^{-1}(x))$$

or equivalently $F(t(x)) = 1 - F(x)$, and hence (1). This argument assumes that t is decreasing (as indeed is (1)); if t were increasing, $F(x) = F(t^{-1}(x))$ for all x implies $t(x) = x$.

The beautiful observation (1), which applies to every continuous distribution F, is certainly very far from new (e.g. Doksum, 1975, MacGillivray, 1986), but seems not to be very widely appreciated. I particularly wish to draw attention to the deep and insightful treatment of this and related issues by Kucerovsky, Marchand & Small (2005). See that paper for a comprehensive analysis additionally allowing transformations with singularities (which I shall touch on briefly in Section 6.1) and much more rigorous and wide-ranging mathematics than will be found here.

Here are some immediate general observations on (1). First, $t(x)$ can be written in terms of the survivor function $\overline{F}(x) = 1 - F(x)$, trivially as $t(x) = F^{-1}\{\overline{F}(x)\}$ and equivalently as $t(x) = \overline{F}^{-1}\{F(x)\}$. The function $t(x)$, as well as being decreasing, is self-inverse: $t(t(x)) = x$; $t^{-1}(x) = t(x)$. And, rearranging a formula above,

$$F(t(x)) + F(x) = 1. \tag{2}$$

Of course, the distribution of $X - t(X)$ is ordinary symmetric about zero.

The symmetry of this subsection, which to make distinct from the different symmetry to follow will be called 'T-symmetry', can be thought of as being on the level of distribution functions and hence intimately associated with transformations of random variables.

2.2 S-Symmetry

In contrast to the distribution-level T-symmetry of Section 2.1, in this subsection I assume the existence of a density function, supposed below to be differentiable except perhaps at its mode, and hence consider symmetry at the level of the density,

$$f_S(s(x)) = f_S(x) \tag{3}$$

for all x and some function $s(x) \neq x$. Motivated by the special case of R-symmetry introduced in the seminal work of Mudholkar & Wang (2007) which takes $s(x) = \psi^2/x$ for $x > 0$, I call the property (3) 'S–symmetry' in Jones (2010, 2012). It is superficially reminiscent of the 'generalized symmetry' of Azzalini (2012) and Azzalini & Capitanio (2014, Section 1.3.2); however, a requirement of the latter that a certain determinant be unity affords no generalization over ordinary symmetry in the univariate case of interest here, but is designed for deployment as a multivariate extension.

Interest in the current paper centres on s being a one-to-one function. If s is decreasing, f_S' has opposite signs at x and $s(x)$ and hence f_S is unimodal or possibly, in the case of finite support, uniantimodal, with mode or antimode at x_0 such that $s(x_0) = x_0$. It also then follows that s, like t in Section 2.1, must be self-inverse. Also as for t, $s(x) \neq x$ increasing is ruled out, in this case because it implies an increasing or decreasing density which would have to take equal values at x and $s(x)$.

As noted in Chaubey, Mudholkar & Jones (2010), the R-symmetric distributions coincide with the Cauchy–Schlömilch distributions introduced by Baker (2008). Similarly, in the unimodal case, the S-symmetric distributions coincide with the 'transformation of scale' distributions of Jones (2010, 2014); see Section 4.4.

3. The S-Symmetric Dual of a Distribution

I will now use f_S specifically for the density of the S-symmetric dual of f defined in the theorem below, that is, the density of an S-symmetric distribution depending only on f and/or F and with

$$s(x) = t(x) = F^{-1}(1 - F(x)). \tag{4}$$

Theorem. *The density of the S-symmetric dual of the distribution F with density f is given by*

$$f_S(x) = \frac{2f(x)}{1 - t'(x)} = \frac{2f(x)f(t(x))}{f(x) + f(t(x))}, \tag{5}$$

where t is given by (4), the latter representation being the harmonic mean of the density $f(x)$ and the function $f(t(x))$.

Proof. The S-symmetry of f_S with s satisfying (4) is obvious from the final expression in (5). That f_S is a density follows from its nonnegativity and the fact that

$$I \equiv \int f_S(x)dx = 2\int \frac{f(x)f(t(x))}{f(x) + f(t(x))}dx = 2\int \left[f(x) - \frac{f^2(x)}{f(x) + f(t(x))}\right] dx.$$

By using the substitution $y = t(x)$ in the second part of the final integral, we get $I = 2 - I$ so that $I = 1$. □

I do not know how to construe and prove uniqueness of the construction above. However, I am not aware of any other candidates for this role.

When the integrals in the proof above have upper limit y, we find that $F_S(y) = 2F(y) - 1 + F_S(t(y))$ where F_S is the distribution function of the dual S-symmetric distribution. Using (2) and rearranging leads to the following intriguing invariance relationship between probabilities of lying in certain intervals under f and f_S:

$$F_S(y) - F_S(t(y)) = F(y) - F(t(y)). \tag{6}$$

By differentiation of f_S, any stationary point, x_p, of it satisfies $\ell(x_p) = \ell(t(x_p))$ where $\ell(y) = f'(y)/f^3(y)$. If f_S is unimodal, this means that its mode, x_0, must equal the median, m_F, of f, because $t(m_F) = F^{-1}(1 - F(m_F)) = m_F$; also, from (5), $f_S(m_F) = f(m_F)$.

Now let $X \sim f$, where $\sim$ denotes 'follows the distribution with density', and $X_S \sim f_S$ given by (5). It can readily be checked that X and X_S are related in the following way:

$$X_S = \begin{cases} X & \text{with probability } \frac{1}{1-t'(X)} = \frac{f(t(X))}{f(X)+f(t(X))}, \\ t(X) & \text{with probability } \frac{-t'(X)}{1-t'(X)} = \frac{f(X)}{f(X)+f(t(X))}, \end{cases} \tag{7}$$

$$X = \begin{cases} X_S & \text{with probability } 1/2, \\ t(X_S) & \text{with probability } 1/2. \end{cases} \tag{8}$$

These relationships are reminiscent of Theorem 2.1 of Jones (2012) and for good reason: see Section 4.4 below. They can be used to good effect in random variate generation of S-symmetric distributions; for the main example thereof, see Section 4.2.

4. Special Cases and Antecedents

4.1 Ordinary Symmetry

If F is ordinary symmetric about θ, $F(2\theta - x) = 1 - F(x)$ and so $t(X)$, which has the same distribution as X, is given by $t(X) = F^{-1}\{F(2\theta - X)\} = 2\theta - X$, as expected. Also, since $f(2\theta - x) = f(x)$, (5) yields $f_S(x) = f(x)$, and there is no distinction between T- and S-symmetry.

4.2 R-Symmetry

If F on $x > 0$ is log-symmetric, $F(\phi^2/x) = 1 - F(x)$ (Seshadri, 1965). By (2), the equidistribution here is of X and $t(X) = \phi^2/X$. Using $f(\phi^2/x) = x^2 f(x)/\phi^2$, the S-symmetric, or more specifically R-symmetric (Mudholkar & Wang, 2007), dual of log-symmetric f has density

$$f_R(x) = 2x^2 f(x)/(x^2 + \phi^2), \qquad x > 0. \tag{9}$$

A particular example of this R-symmetric duality is when f is the density of $1/\sqrt{B}$ and B follows the Birnbaum-Saunders distribution, which is dual to the R-symmetric root-reciprocal inverse Gaussian, or CoGaussian, distribution (Mudholkar, Yu & Awadalla, 2014), that is, the distribution of $1/\sqrt{G}$ when G follows the inverse Gaussian distribution. In Jones (2012), I mention this and the consequential relationships between the Birnbaum-Saunders and inverse Gaussian distributions themselves, which give a derivation of the Michael, Schucany & Haas (1976) method for generating inverse Gaussian random variates.

The lognormal distribution, in its usual normal-based parametrisation, is both log-symmetric (about e^{μ}) and R-symmetric (about $e^{\mu-\sigma^2}$). However, the lognormal distribution is not 'self-dual', and it is clear from (9) that no distribution can be.

4.3 Exponential and Power Law Symmetries

If F is the exponential distribution with density $f(x) = \lambda e^{-\lambda x}$, $\lambda, x > 0$, then it is easy to show that $t(x) = -\log(1 - e^{-\lambda x})/\lambda \equiv t_e(x)$, so if X has the exponential distribution, $t_e(X)$ has the same exponential distribution. Changing argument in (2), exponential T-symmetry corresponds to distributions F on $x > 0$ such that

$$F(-\lambda \log u) + F(-\lambda \log(1-u)) = 1,$$

$0 < u < 1$. The S-symmetric dual of the exponential distribution has density

$$2\lambda e^{-\lambda x}(1 - e^{-\lambda x}), \qquad \lambda, x > 0.$$

This is a special case of the exponentiated exponential distribution (Gupta & Kundu, 1999). Use of the exponential symmetry represented by $t_e(x)$ in transformation of scale distributions is suggested from other considerations in Jones (2010, 2014).

If F is the power law distribution with density $f(x) = \alpha x^{\alpha-1}$, $\alpha > 0$, $0 < x < 1$, then $t_p(X) \equiv (1 - X^{\alpha})^{1/\alpha}$ follows the same power law distribution. In this case, T-symmetry corresponds to distributions on $0 < x < 1$ such that $F(x^{\alpha})+F((1-x)^{\alpha}) = 1$. The S-symmetric dual of this F has a more complicated density that is omitted.

4.4 Transforming Ordinary Symmetric Distributions

Write $F(x) = G(\Pi^{-1}(x))$ where G is an arbitrary distribution ordinary symmetric about zero and $\Pi = F^{-1}(G)$ is the appropriate increasing function such that $\Pi(Y)$ follows the distribution F when Y comes from G (with density g). Thus,

$$t(x) = \Pi\left(G^{-1}\left\{1 - G(\Pi^{-1}(x))\right\}\right) = \Pi\left(G^{-1}\left\{G(-\Pi^{-1}(x))\right\}\right) = \Pi(-\Pi^{-1}(x)), \quad (10)$$

using the ordinary symmetry about zero of G. Now, since $f(x) = g(\Pi^{-1}(x))/\Pi'(\Pi^{-1}(x))$ and $t'(x) = -\Pi'(-\Pi^{-1}(x))/\Pi'(\Pi^{-1}(x))$, the density of the S-symmetric dual of F is

$$\frac{2g(\Pi^{-1}(x))}{\Pi'(\Pi^{-1})(x) + \Pi'(-\Pi^{-1})(x)}.$$

The particularly simple form $f_S(x) = 2g(\Pi^{-1}(x))$ arises if Π is chosen to satisfy

$$\Pi'(y) + \Pi'(-y) = 1 \text{ or essentially equivalently } \Pi(y) - \Pi(-y) = y. \qquad (11)$$

These are precisely the 'transformation of scale' distributions of Jones (2010, 2014). And they are dual to F written as $G(\Pi^{-1}(x))$ using the same G and Π.

In fact, because of the arbitrary nature of the choice of symmetric g, for any $t(x)$ given by (1), it is possible to choose Π through a variation on the right-hand equation in (11), namely

$$\Pi^{-1}(x) = x - t(x),$$

to equate the class of S-symmetric distributions to the class of transformation of scale distributions. Since $\Pi^{-1} = G^{-1}(F)$, this is equivalent to specifying G via the following ordinary symmetrisation of F:

$$G^{-1}(u) = F^{-1}(u) - F^{-1}(1-u).$$

5. Further Aspects of T- and S-Symmetry Together

5.1 Measures of Ordinary Asymmetry

Ordinary asymmetry, that is, the degree to which a distribution departs from ordinary symmetry, might be measured by how far the symmetry transformation $F^{-1}(1-F(x))$ departs from $-x$. Doksum's (1975) 'symmetry function' is of precisely this form. It is $\mathcal{A}_D(x) = \frac{1}{2}\{F^{-1}(1-F(x)) + x\}$. Doksum argues that $\mathcal{A}_D(x)$ should be compared with the natural centre of ordinary symmetry, the median m_F, yielding a functional asymmetry measure proportional to $F^{-1}(1-F(x)) - 2m_F + x$. Further, making this quantity scale free by dividing by the corresponding scale measure $F^{-1}(1-F(x)) - x$ yields the function

$$\gamma_F(x) \equiv \frac{F^{-1}(1-F(x)) - 2m_F + x}{F^{-1}(1-F(x)) - x}. \tag{12}$$

This differs from the more usual quantile-based asymmetry function of David & Johnson (1956),

$$\gamma_F(u) \equiv \frac{F^{-1}(1-u) - 2m_F + F^{-1}(u)}{F^{-1}(1-u) - F^{-1}(u)}, \quad 0 < u < 1, \tag{13}$$

only by the change of scale $u = F(x)$. See also MacGillivray (1986).

On the other hand, Critchley & Jones (2008) propose the following density-based asymmetry function for use with unimodal distributions (see also Boshnakov, 2007). This is of particular interest for unimodal S-symmetric distributions. Write $x_L(p)$ and $x_R(p)$, $0 < p < 1$, for the solutions of $f_S(x) = pf_S(x_0)$ to the left and right of the mode, x_0, respectively, when f_S is unimodal; note that $x_L(p) = s(x_R(p))$. Then, their scaled asymmetry function takes the form

$$\gamma_{f_S}(p) \equiv \frac{x_L(p) - 2x_0 + x_R(p)}{x_L(p) - x_R(p)} = \frac{s(x_R(p)) - 2x_0 + x_R(p)}{s(x_R(p)) - x_R(p)}. \tag{14}$$

For the S-symmetric dual of f, $s(x) = F^{-1}(1 - F(x))$, $x_0 = m_F$, and if we set $x = x_R(p)$, the resulting density-based measure $\gamma_{f_S}(x)$ obtained from (14) is the same as $\gamma_F(x)$ at (12). The versions of (12) based on a change of scale to $(0, 1)$ differ only in the way this is done: distribution-based $u = F(x)$ in (13) and density-based $p = f_S(x)/f_S(m_F)$ in (14).

5.2 An Intermediate Distribution

With t given by (4), from (7), $t(X_S)$ is like X_S with the probabilities reversed:

$$t(X_S) = \begin{cases} X & \text{with probability } \frac{-t'(X)}{1-t'(X)} = \frac{f(X)}{f(X)+f(t(X))}, \\ t(X) & \text{with probability } \frac{1}{1-t'(X)} = \frac{f(t(X))}{f(X)+f(t(X))}. \end{cases} \tag{15}$$

The distribution of $t(X_S)$ has density

$$f_{T(S)}(x) = \frac{2f^2(x)}{f(x) + f(t(x))}, \tag{16}$$

which features in the proof of the theorem in Section 3. As with (5), it is superficially surprising that this is a bona fide density.

6. Further Aspects of T-Symmetry Alone

6.1 Two Equi-Distributed Transformations?

The standard Cauchy distribution provides a special case in which X has the same distribution as both $-X$ and $1/X$. Since $F(x) = (1/2) + (\tan^{-1} x)/\pi$, (1) gives $t(x) = -x$.

The equi-distribution of X and $1/X$ arises by allowing transformations with singularities (for much more on the latter, see Kucerovsky, Marchand & Small, 2005). Equi-distribution of X and $-X$ remains unique among continuous decreasing transformations. Another aspect of this is that $1/X$ is the unique continuous decreasing, and hence equi-distributed, transformation for the half-Cauchy distribution (see also Seshadri, 1965). This is the standard Cauchy distribution truncated at 0; it has $F(x) = 2(\tan^{-1} x)/\pi$, for which (1) gives $t(x) = \tan\{(\pi/2) - \tan^{-1} x\} = 1/x$.

6.2 Survival Copulas

Suppose that $X_1 \sim f_1$ with distribution function F_1 and $X_2 \sim f_2$ with distribution function F_2 are jointly distributed with distribution function $F(x_1, x_2)$. Of course, the joint distribution of $U_1 = F_1(X_1)$ and $U_2 = F_2(X_2)$ is the copula, C, associated with F. Now, a well known alternative copula associated with this copula is its survival copula, $\hat{C}(v_1, v_2) = v_1 + v_2 - 1 + C(1 - v_1, 1 - v_2)$, which is the distribution function of $1-U_1$ and $1-U_2$ (e.g. Nelsen, 2006, Section 2.6). The name, of course, arises because this copula is the distribution of $V_1 = \overline{F}_1(X_1)$ and $V_2 = \overline{F}_2(X_2)$ which, in the current context, can also be seen as $V_i = F_i(t_i(X_i))$ where $t_i(x) = F_i^{-1}(1-F_i(x))$, $i = 1, 2$, as at (1).

The distribution function associated with the survival copula is simply $F(t_1(x), t_2(y))$. Its marginal distributions are F_1 and F_2, the same as those of F, but its dependence structure is that of $t_1(X_1)$ and $t_2(X_2)$ rather than X_1 and X_2.

6.3 Two Ways From X to Z

Suppose that $X \sim f$ and $Z \sim \ell$ with distribution function L, say. Then the usual way of transforming Z to get X, via the PIT, is $Z = L^{-1}\{F(X)\}$. This can equivalently be written $Z = \overline{L}^{-1}\{\overline{F}(X)\}$. However, again, since $F(X)$ and $1-F(X)$ have the same distribution, this transformation is not, as might be assumed, unique: there is another pair of equivalent transformations with the same distribution as Z, namely, $L^{-1}\{\overline{F}(X)\} = \overline{L}^{-1}\{F(X)\}$. Of course, the latter pair are the result of applying the transformation $L^{-1}(\overline{L})$ to the former pair.

As just one minor example, the Weibull distribution arises as the distribution of E^{β}, say, where E follows the exponential distribution with parameter $\lambda > 0$, and $\beta > 0$. What is not so well appreciated is that the Weibull distribution also arises as the distribution of $\{-\log(1-e^{-\lambda E})/\lambda\}^{\beta}$.

7. Closing Remarks

To repeat, T-symmetry is a property of every univariate continuous distribution, and it and its consequences may well be familiar to many readers. S-symmetry, on the other hand, defines a particular class of distributions, those with the density symmetry property (3) for some function $s(x) \neq x$. The main aim of this article — the theorem of Section 3 — has been to introduce the S-symmetric analogue of any given (T-symmetric) distribution F and to explore the consequences thereof, chief amongst which may be the random variable connections at the end of Section 3 and the relationships between measures of ordinary symmetry in Section 5.1.

References

Azzalini, A., 2012. Selection models under generalized symmetry settings. Ann. Inst. Statist. Math. 64, 737–750.

Azzalini, A., with the collaboration of Capitanio, A. 2014. *The Skew-Normal and Related Families*. Cambridge University Press.

Baker, R., 2008. Probabilistic applications of the Schlömilch transformation. Commun. Statist. Theor. Meth. 37, 2162–2176.

Boshnakov, G.N., 2007. Some measures for asymmetry of distributions. Statist. Probab. Lett. 77, 1111-1116.

Chaubey, Y.P., Mudholkar, G.S., Jones, M.C., 2010. Reciprocal symmetry, unimodality and Khintchine's theorem. Proc. Roy. Soc. Ser. A 466, 2079–2096.

Critchley, F., Jones, M.C., 2008. Asymmetry and gradient asymmetry functions: density-based skewness and kurtosis. Scand. J. Statist. 35, 425–437.

David, F.N., Johnson, N.L., 1956. Some test of significance with ordered variables. J. Roy. Statist. Soc. Ser. B 18, 1–20.

Doksum, K.A., 1975. Measures of location and asymmetry. Scand. J. Statist. 2, 11–22.

Gupta, R.D., Kundu, D., 1999. Generalized exponential distributions. Austral. New Z. J. Statist. 41, 173–188.

Jones, M.C., 2010. Distributions generated by transformations of scale using an extended Schlömilch transformation. Sankhyā Ser. A 72, 359–375.

Jones, M.C., 2012. Relationships between distributions with certain symmetries. Statist. Probab. Lett. 82, 1737–1744.

Jones, M.C., 2014. Generating distributions by transformation of scale. Statist. Sinica, 24, 749–771.

Kucerovsky, D., Marchand, E., Small, R.D., 2005. On the equality in distribution of the random variables X and $g(X)$. Internat. J. Pure Appl. Math., 23, 93–114.

MacGillivray, H.L., 1986. Skewness and asymmetry: measures and orderings. Ann. Statist. 14, 994–1011.

Michael, J.R., Schucany, W.R., Haas, R.W., 1976. Generating random variates using transformations with multiple roots. Amer. Statist. 30, 88–90.

Mudholkar, G.S., Wang, H., 2007. IG-symmetry and R-symmetry: inter-relations and applications to the inverse Gaussian theory. J. Statist. Planning Inference 137, 3655–3671.

Mudholkar, G.S., Yu, Z., Awadalla, S.A., 2014. The CoGaussian distribution: a model for right skewed data. Manuscript.

Nelsen, R.B., 2006. *An Introduction to Copulas*, second edition. Springer.

Seshadri, V., 1965. On random variables which have the same distribution as their reciprocals. Canad. Math. Bull. 8, 819–824.