

Families of complementary distributions

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ABSTRACT

Each continuous distribution on $(0, 1)$, with cumulative distribution function F say, has a complementary distribution which is the distribution with cumulative distribution function F^{-1} . Some basic general properties of complementary distributions are given. A particular focus of this article is then the construction of families of distributions, each based on a given F and indexed by a single additional parameter, which are closed under complementarity; properties of these families of distributions are explored, as are those of a particular special case.

Keywords:

Complementary beta distribution

Distributions on $(0, 1)$

1. Introduction

This article is concerned with understanding certain theoretical structures underlying continuous distributions for modelling data on a known finite interval which, without loss of generality, can be taken to be $(0, 1)$. Consider such a distribution with cumulative distribution function (c.d.f.) F , say. It is then the case that F^{-1} is also the c.d.f. of a distribution on $(0, 1)$.

Definition 1.1. The distribution with c.d.f. F^{-1} will be termed the *complementary distribution* to F .

See Jones (2002) for exploration of this notion in the special case where F is the c.d.f. of a beta distribution, and the distribution with c.d.f. F^{-1} was called the complementary beta distribution. Some basic general properties of complementary distributions are given in Section 2 of the current article, partly by way of background to the remainder of the article, and almost entirely for the first time.

A particular focus of this article is the construction of families of distributions, each based on a given F and indexed by a single additional parameter $0 \leq p \leq 1$, which are closed under complementarity: the family includes F , F^{-1} (and, as it happens, \mathcal{U} , the c.d.f. of the uniform distribution, which is its own complement) and all distributions “between F and F^{-1} ” in a certain sense, including their complementary distributions: members of the family indexed by p are complementary to members indexed by $1 - p$. Details of these families of complementary distributions and their properties are given in Section 3. By way of example, one particular tractable family of distributions with monotone densities is the subject of Section 4. The article closes with further remarks in Section 5.

2. Complementary distributions

Let V be a random variable following a distribution with c.d.f. F ; write this as $V \sim F$. Then, defining $W = F(F(V))$, it is easy to show that $W \sim F^{-1}$. Equivalently, if $U \sim \mathcal{U}$, then $W = F(U) \sim F^{-1}$, which typically affords straightforward random variate generation from F^{-1} ; contrast this with $V = F^{-1}(U) \sim F$.

Write f for the probability density function (p.d.f.) associated with F . The p.d.f. associated with F^{-1} is $1/f(F^{-1}(w))$, which is the quantile density function, q , of F . A complementary pair of distributions therefore exhibits reciprocal behaviour at the extremes of the unit interval: $\lim_{w \rightarrow 0 \text{ or } 1} q(w) = 1/\lim_{v \rightarrow 0 \text{ or } 1} f(v)$. Since $q'(w) = -f'(F^{-1}(w))/f^3(F^{-1}(w))$, f' and q' exhibit the same number of zeroes. Zeroes of f' , situated at v_i , say, correspond to zeroes of q' situated at $w_i = F(v_i)$. In particular, if f is decreasing (resp. increasing), then q is increasing (resp. decreasing). If f is unimodal (resp. uniantimodal) with mode (resp. antimode) at $0 < v_0 < 1$, then q is uniantimodal (resp. unimodal) with antimode (resp. mode) at $w_0 = F(v_0)$. If F is a

distribution symmetric about $v = 1/2$ then so is F^{-1} . The unique distribution whose complementary distribution is the same as the original distribution is the uniform distribution.

Moments of complementary distributions can be written as

$$E_{F^{-1}}(W^r) = \int_0^1 F^r(v)dv.$$

In particular, as one might expect,

$$E_{F^{-1}}(W) = 1 - E_F(V). \quad (1.1)$$

It also turns out that the variance of the complementary distribution satisfies

$$V_{F^{-1}}(W) = 2E_F(V) - \{E_F(V)\}^2 - 2E_F\{VF(V)\} \quad (1.2)$$

while

$$\begin{aligned} S_{F^{-1}}(W) = & -3E_F\{VF^2(V)\} + 6\{E_F(VF(V))\}\{1 - E_F(V)\} \\ & + \{E_F(V)\}^3 + 3E_F(V)\{2E_F(V) - E_F(V^2)\} - 3E_F(V) \end{aligned} \quad (1.3)$$

where, generically, $S_H(T) = E_H\{T - E_H(T)\}^3$ denotes the numerator of the classical skewness measure when $T \sim H$. Direct proofs of these assertions are omitted as they arise as special cases of more general formulas provided in Section 3.

As noted in Jones (2002), integrals of the form $\int F^r(x)(1 - F)^s(x)dx$ are central components in the calculation of expectations of order statistics of a distribution with c.d.f. F , and associated quantities such as expectations of spacings and L-moments. In the case of the complementary distribution with c.d.f. F^{-1} this integral equates simply to $E_F\{V^r(1 - V)^s\}$.

As for L-moments themselves (Hosking, 1990), which I generically denote $L_{n;H}(T)$: $L_{1;H}(T)$ is the mean of H ; the scale measure $L_{2;H}(T)$ is one-half of Gini's mean difference, which for the original and complementary distribution satisfies

$$L_{2;F}(V) = 2E_F\{VF(V)\} - E_F(V), \quad L_{2;F^{-1}}(W) = E_F(V) - E_F(V^2); \quad (1.4)$$

and $L_{3;H}(T)$, the numerator of the L-skewness measure, satisfies

$$\begin{aligned} L_{3;F}(V) &= 6E_F\{VF^2(V)\} - 6E_F\{VF(V)\} + E_F(V), \\ L_{3;F^{-1}}(W) &= 3E_F(V^2) - 2E_F(V^3) - E_F(V). \end{aligned} \quad (1.5)$$

The statements concerning L-moments of both F and F^{-1} can be readily obtained directly as well as being special cases of results in Section 3.

Combining (1.2), (1.4) and the standard formula for $V_F(V)$, the following remarkable invariance property under complementarity emerges which can equivalently be written as the equality of scale differences between F and F^{-1} measured in two different ways.

THEOREM 2.1

$$V_{F^{-1}}(W) - L_{2;F^{-1}}(W) = V_F(V) - L_{2;F}(V)$$

or equivalently

$$V_{F^{-1}}(W) - V_F(V) = L_{2;F^{-1}}(W) - L_{2;F}(V).$$

3. Families of complementary distributions

Repurposing an approach to the combination of certain functions and their inverses taken by Jones & Pewsey (2012, Section 3), the following definition is made.

Definition 3.1. The c.d.f. of the new family of distributions is given by

$$G_p(u) = \{(1-p)u + (2p-1)M_p^{-1}(u)\} / p, \quad 0 < u < 1,$$

where the parameter p takes values in $[0, 1]$ and

$$M_p(u) = pF(u) + (1-p)u$$

is the c.d.f. of the usual $p : 1 - p$ mixture of F and \mathcal{U} .

It is immediate that

$$G_{1/2}(u) = u = \mathcal{U}(u) \quad \text{and} \quad G_1(u) = F^{-1}(u).$$

Defining $G_0(u)$ by continuity in p and noting that

$$[u + p\{F(u) - u\}]^{-1} = u - p\{F(u) - u\} + o(p)$$

as $p \rightarrow 0$ shows that

$$G_0(u) = F(u).$$

So, as p increases, the family moves from F through \mathcal{U} to F^{-1} .

As suggested in Section 1, the appeal of this formulation lies in the fact that, for all $0 \leq p \leq 1$, G_p and G_{1-p} are complementary to one another (this is clearly so when $p = 0$ or 1 and, indeed, when $p = 1/2$).

THEOREM 3.1

$$G_p(u) = G_{1-p}^{-1}(u).$$

Proof

I need to prove the result for any $p \in (0, 1/2) \cup (1/2, 1)$. Observe that

$$\begin{aligned} G_p(M_p(u)) &= \{(1-p)M_p(u) + (2p-1)u\}/p \\ &= \{(1-p)^2u + p(1-p)F(u) + (2p-1)u\}/p \\ &= pu + (1-p)F(u) = M_{1-p}(u). \end{aligned}$$

So, $G_p(u) = M_{1-p}(M_p^{-1}(u))$ which implies that

$$G_p^{-1}(u) = M_p(M_{1-p}^{-1}(u)) = G_{1-p}(u).$$

COROLLARY 3.1

For $0 < \alpha < 1$, the quantile function associated with G_p is

$$G_p^{-1}(\alpha) = G_{1-p}(\alpha) = \{p\alpha + (1-2p)M_{1-p}^{-1}(\alpha)\}/(1-p).$$

If F is symmetric about $v = 1/2$, then so is M_p and so is every G_p derived from F . In that case, of course, $G_p^{-1}(1/2) = 1/2$.

Suppose that it is desired to generate $X \sim G_p$. As an alternative to numerically inverting M_{1-p} in the course of implementing the probability integral transformation using Corollary 3.1, random variate generation can also be performed via the distribution of $Y = M_p^{-1}(X)$ which has c.d.f. $G_p(M_p(y)) = M_{1-p}(y)$ (the equality being central to the proof of Theorem 3.1 above). If $V \sim F$ and independent $U \sim \mathcal{U}$ are available, then choose

$$Y = \begin{cases} U & \text{with probability } p, \\ V & \text{with probability } 1-p \end{cases}$$

and, finally, set $X = M_p(Y) = pF(Y) + (1-p)Y$.

Corollary 3.2 lists some distributional relationships associated with G_p . Each is the specialisation to G_p of a general result: the first of a general property of complementary distributions mentioned in Section 2; the second of the fact that if $V \sim F$ then $F^{-1}(1-F(V)) \sim F$ also; and the third of a general result of Marchand, Jones & Strawderman (2018).

COROLLARY 3.2

$$\begin{aligned} X \sim G_p &\Rightarrow G_p(G_p(X)) \sim G_{1-p}, \\ X \sim G_p &\Rightarrow G_{1-p}(1-G_p(X)) \sim G_p, \\ U \sim \mathcal{U} &\Rightarrow G_p(U) - U \sim U - G_{1-p}(U). \end{aligned}$$

Since $G_p(u) = M_{1-p}(M_p^{-1}(u))$, the density associated with G_p is

$$\begin{aligned} g_p(u) &= m_{1-p}(M_p^{-1}(u))/m_p(M_p^{-1}(u)) \\ &= \{p + (1-p)f(M_p^{-1}(u))\}/\{1-p + pf(M_p^{-1}(u))\} \end{aligned}$$

where f and m_p are the densities associated with F and M_p , respectively.

THEOREM 3.2

When $0 \leq p < 1/2$, g_p shares its modality with $F' = f$; when $1/2 < p \leq 1$, g_p shares its modality with $(F^{-1})' = 1/f(F^{-1})$. Moreover, if any mode and/or antimode of f or $1/f(F^{-1})$ is situated at v_0 say, then the corresponding mode and/or antimode of g_p is situated at $M_p(v_0)$. This means that said mode/antimode lies between v_0 and $F(v_0)$.

Proof

It is easy to show that $g_p'(u)$ comprises positive terms multiplied by $(1-2p)f'(M_p^{-1}(u))$, from which the theorem follows.

Notice that this is quite different from the situation with a $p : 1-p$ mixture of F^{-1} and F which typically adds modes/antimodes; the current construction does not.

Tail behaviour of g_p also follows immediately from that of f , for any $0 \leq p \leq 1$:

THEOREM 3.3

Let either endpoint of the support, 0 or 1, be denoted u_e and let c be a finite positive constant. Then

$$\begin{aligned} f(u_e) \rightarrow 0 &\Rightarrow g_p(u_e) \rightarrow p/(1-p), \\ f(u_e) \rightarrow c &\Rightarrow g_p(u_e) \rightarrow \{p + (1-p)c\}/(1-p+pc), \\ f(u_e) \rightarrow \infty &\Rightarrow g_p(u_e) \rightarrow (1-p)/p. \end{aligned}$$

3.1 Moments and L-moments

I now turn attention to moments (and then to L-moments). Making use of the facts that, for any random variable T on $(0, 1)$ with c.d.f. H , say,

$$E_H(T) = 1 - \int_0^1 H(t) dt = \int_0^1 H^{-1}(t) dt,$$

the following is true:

THEOREM 3.4

$$E_{G_p}(X) = p - (2p - 1)E_F(V)$$

Proof

$$\begin{aligned} E_{G_p}(X) &= 1 - \int_0^1 G_p(x)dx \\ &= 1 - \{(1 - p)/2 + (2p - 1)E_{M_p}(S)\} / p \\ &= 1 - [(1 - p)/2 + (2p - 1)\{(1 - p)/2 + pE_F(V)\}] / p \end{aligned}$$

which reduces to the statement of the theorem.

Combining Theorem 3.4 with (1.1) yields the following corollary.

COROLLARY 3.3

$$E_{G_p}(X) = pE_{F^{-1}}(W) + (1 - p)E_F(V)$$

Note that the mean of G_p is the same as the mean of a $p : 1 - p$ mixture of F^{-1} and F . Both vary linearly in p starting from $E_F(V)$ and ending at $E_{F^{-1}}(W)$.

I continue on to consider the second and third moments associated with G_p and then the second and third L-moments. In so doing, proofs of results are relegated to the Appendix.

THEOREM 3.5

$$\begin{aligned} V_{G_p}(X) &= p(1 - p)/3 - (2p - 1) \times \\ &\quad [2pE_F\{VF(V)\} - 2pE_F(V) + (1 - p)V_F(V) + p\{E_F(V)\}^2] \end{aligned}$$

Combining Theorem 3.5 with (1.2) yields the following corollary.

COROLLARY 3.4

$$V_{G_p}(X) = p(1 - p)/3 + (2p - 1) \{pV_{F^{-1}}(W) - (1 - p)V_F(V)\}$$

Notice that V_{G_p} is, as you might hope, a quadratic function of p going from $V_F(V)$ when $p = 0$ through $1/12$ when $p = 1/2$ to $V_{F^{-1}}(W)$ when $p = 1$ (in increasing, decreasing, unimodal, or uniantimodal fashion as determined by the relative values of $V_F(V)$, $1/12$ and $V_{F^{-1}}(W)$). The variance of G_p is of a quite different character

from the variance of a $p : 1 - p$ mixture of F^{-1} and F (which does not include the uniform distribution). It is

$$V_{\text{mix}} = pV_{F^{-1}}(W) + (1 - p)V_F(V) + p(1 - p)\{E_{F^{-1}}(W) - E_F(V)\}^2,$$

which depends on the means as well as the variances associated with F^{-1} and F .

It is worth a look at the skewness too. I will focus on $S_{G_p}(X) = E_{G_p}\{X - E_{G_p}(X)\}^3$ while recognising that the (classical) skewness of is actually, of course, $S_{G_p}(X)/V_{G_p}^{3/2}(X)$.

THEOREM 3.6

$$\begin{aligned} S_{G_p}(X) = & (2p - 1) \times [-p(1 - p)/4 - 3p^2 E_F\{VF^2(V)\} - 3p(1 - p)E_F\{V^2F(V)\} \\ & + 6p^2 E_F\{VF(V)\} - 6p(2p - 1)E_F\{VF(V)\} E_F(V) \\ & - (1 - p)^2 S_F(V) - p(3p - 1)\{E_F(V)\}^3 \\ & + 3p(1 - p)V_F(V) - 3p(1 - p)V_F(V) E_F(V) \\ & + 3p(3p - 1)\{E_F(V)\}^2 + p(1 - 4p)E_F(V)]. \end{aligned}$$

Combining Theorem 3.6 with (1.3) and the usual formula for $S_F(V)$ yields the following corollary.

COROLLARY 3.5

$$\begin{aligned} S_{G_p}(X) = & (2p - 1) \times (p^2 S_{F^{-1}}(R) - (1 - p)^2 S_F(V) + p(1 - p) \times \\ & [6E\{VF(V)\} - 3E\{V^2F(V)\} + \{E_F(V)\}^3 \\ & + 3V_F(V)\{1 - E_F(V)\} - 3\{E_F(V)\}^2 + E_F(V) - 1/4]) \end{aligned}$$

It is the case that S_{G_p} is a cubic function of p going from $S_F(V)$ when $p = 0$ through 0 when $p = 1/2$ to $S_{F^{-1}}(W)$ when $p = 1$. There is an extra ‘degree-of-freedom’ in the shape of S_{G_p} as a function of p , however, compared with that of V_{G_p} .

Turning to L-moments, I first compute the second L-moment (the first being the mean), namely the scale measure equal to one half of Gini’s mean difference. In derivations (see the Appendix), I will directly use the generic formula

$$L_{2;H}(T) = \int_0^1 H(t)\{1 - H(t)\} dt.$$

THEOREM 3.7

$$\begin{aligned} L_{2;G_p}(X) = & 2p(1 - p)/3 - (2p - 1) \times \\ & [2(1 - p)E_F\{VF(V)\} - E_F(V) + pE_F(V^2)]. \end{aligned}$$

The following intriguing corollary follows by combining Theorem 3.7 with (1.4).

COROLLARY 3.6

$$L_{2;G_p}(X) = 2p(1-p)/3 + (2p-1) \{pL_{2;F^{-1}}(W) - (1-p)L_{2;F}(V)\}.$$

The parallels between results for the variance (Theorem 3.5, Corollary 3.4) and those for the second L-moment (Theorem 3.7, Corollary 3.6) are very strong. $L_{2;G_p}$ is also, as you might hope, a quadratic function of p going from $L_{2;F}$ when $p = 0$ through $1/6$ when $p = 1/2$ to $L_{2;F^{-1}}$ when $p = 1$ (in increasing, decreasing, unimodal, or uniantimodal fashion as determined by the relative values of $L_{2;F}$, $1/6$ and $L_{2;F^{-1}}$).

Next, it's the turn of a second skewness measure, the third L-moment

$$L_{3;H}(T) = \int_0^1 H(t)\{1-H(t)\}\{2H(t)-1\} dt = \int_0^1 \{3H^2(t) - 2H^3(t) - H(t)\} dt.$$

(The L-skewness of G_p is actually $L_{3;G_p}(X)/L_{2;G_p}(X)$.)

THEOREM 3.8

$$L_{3;G_p}(X) = (2p-1) \times [-p(1-p)/2 + 6(1-p)E_F\{VF(V)\} + 3pE_F(V^2) - 2p^2E_F(V^3) - 6(1-p)^2E_F\{VF^2(V)\} - 6p(1-p)E_F\{V^2F(V)\} - E_F(V)]$$

Combining Theorem 3.8 with (1.5) yields the final corollary in this subsection.

COROLLARY 3.7

$$L_{3;G_p}(X) = (2p-1) \times (p^2L_{3;F^{-1}}(W) - (1-p)^2L_{3;F}(V) + p(1-p) \times [6E_F\{VF(V)\} - 6E\{V^2F(V)\} + 3E_F(V^2) - 2E_F(V) - 1/2]).$$

The parallels between results for the classical skewness (Theorem 3.6, Corollary 3.5) and those for the third L-moment (Theorem 3.8, Corollary 3.7) are very strong too.

3.2 Equivalent formulations of G_p

Additionally define

$$N_p(u) = pu + (1-p)F^{-1}(u) = M_p(F^{-1}(u))$$

to be the c.d.f. of the usual $p : 1 - p$ mixture of \mathcal{U} and F^{-1} . Then the following lemma is true: to see this, simply expand the left-hand side of $M_p(M_p^{-1}(u)) = u$.

LEMMA 3.1

$$pN_p^{-1}(u) + (1 - p)M_p^{-1}(u) = u.$$

It follows that G_p can be written in the several equivalent ways in Corollary 3.8 to follow. The first of these is Definition 3.1 in terms of \mathcal{U} and F , the second is an equivalent definition in terms of \mathcal{U} and F^{-1} , and the third an equivalent, ‘more symmetric’, definition in terms of \mathcal{U} , F and F^{-1} .

COROLLARY 3.8

$$\begin{aligned} G_p(u) &= \{(1 - p)u + (2p - 1)M_p^{-1}(u)\} / p \\ &= \{pu - (2p - 1)N_p^{-1}(u)\} / (1 - p) \\ &= pM_p^{-1}(u) + (1 - p)N_p^{-1}(u) \\ &= M_{1-p}(M_p^{-1}(u)) = N_{1-p}(N_p^{-1}(u)). \end{aligned} \tag{3.1}$$

In particular, formulation (3.1) shows G_p to be interpretable as the c.d.f. of the usual $p : 1 - p$ mixture of the complementary distributions of two distributions that are themselves $p : 1 - p$ mixtures, namely the $p : 1 - p$ mixture of F and \mathcal{U} and the $p : 1 - p$ mixture of \mathcal{U} and F^{-1} .

4. Example: a tractable family of complementary distributions

Arguably the simplest one-parameter choice for F is

$$F(u) = F(u; \ell) = \ell u^2 + (1 - \ell)u, \quad 0 < u < 1,$$

with parameter $\ell \in [-1, 1]$: this has linear densities, increasing (constant) decreasing as $\ell > (=) < 0$; moreover, $F(u; -\ell) = 1 - F(1 - u; \ell)$. This is the family of distributions on $(0, 1)$ corresponding to the ‘quadratic rank transmutation map’ of Shaw & Buckley (2009) (and hence underlying the ‘transmuted distributions’ of numerous more recent papers e.g. Bourguignon, Ghosh & Cordeiro, 2016); see Figure 1 for some of their densities.

Note that the complementary transmutation map is

$$F^{-1}(u) = \left\{ \sqrt{4\ell u + (1 - \ell)^2} + (\ell - 1) \right\} / 2\ell$$

and that

$$M_p(u) = p\ell u^2 + (1 - p\ell)u = F(u; p\ell).$$

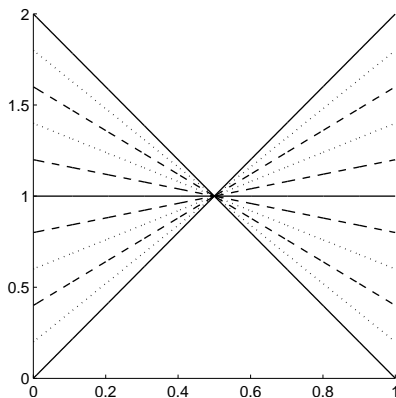


Figure 1: Densities $f(u; \ell) = 2\ell u + 1 - \ell$ plotted for $\ell = -1(0.2)1$ using different line-styles for clarity: towards the left, higher curves correspond to smaller ℓ .

Utilising this choice for F in G_p results in the two-parameter family of distributions with

$$G_p(u; \ell) = \left\{ (2p - 1)\sqrt{4p\ell u + (1 - p\ell)^2} + 2(1 - p)p\ell u + (2p - 1)(p\ell - 1) \right\} / 2p^2\ell.$$

This distribution includes all the F 's, when $p = 0$, all the F^{-1} 's, when $p = 1$, and many distributions 'in between' (including the uniform when $p = 1/2$). The distribution inherits from F the property that $G_p(u; -\ell) = 1 - G_p(1 - u; \ell)$. These cdfs are plotted for a range of values of p in Figure 2(a) when $\ell = 1$; they correspond to $F(u) = u^2$, $F^{-1}(u) = \sqrt{u}$.

The densities of this family of distributions are

$$g_p(u; \ell) = \left\{ 1 - p + (2p - 1)/\sqrt{4p\ell u + (1 - p\ell)^2} \right\} / p.$$

They are necessarily monotone because F always is: if $f(u; \ell)$ increases (decreases) [that is, $\ell > 0$ ($\ell < 0$)] then $g_p(u; \ell)$ increases (decreases) when $0 < p < 1/2$ ($1/2 < p < 1$) (as is also clear directly). As $u \rightarrow 0$, $g_p(u; \ell) \rightarrow (1 - \ell + \ell p)/(1 - \ell p)$ while as $u \rightarrow 1$, $g_p(u; \ell) \rightarrow (1 + \ell - \ell p)/(1 + \ell p)$. The pdfs corresponding to the cdfs in Figure 2(a) are shown in Figure 2(b). (It is worth stressing again that the densities $g_p(1 - u; \ell)$ are also in the family since they equate to $g_p(u; -\ell)$.)

Because $E_{F(\cdot; \ell)}(V) = (3 + \ell)/6$, it is found that

$$E_{G_p(\cdot; \ell)}(X) = 1/2 - (2p - 1)\ell/6.$$

Also, because $V_{F(\cdot; \ell)}(V) = (3 - \ell^2)/36$ and $E_{F(\cdot; \ell)}\{VF(V; \ell)\} = (20 + 5\ell - \ell^2)/60$,

$$V_{G_p(\cdot; \ell)}(X) = 1/12 - (2p - 1)(4p - 5)\ell^2/180$$

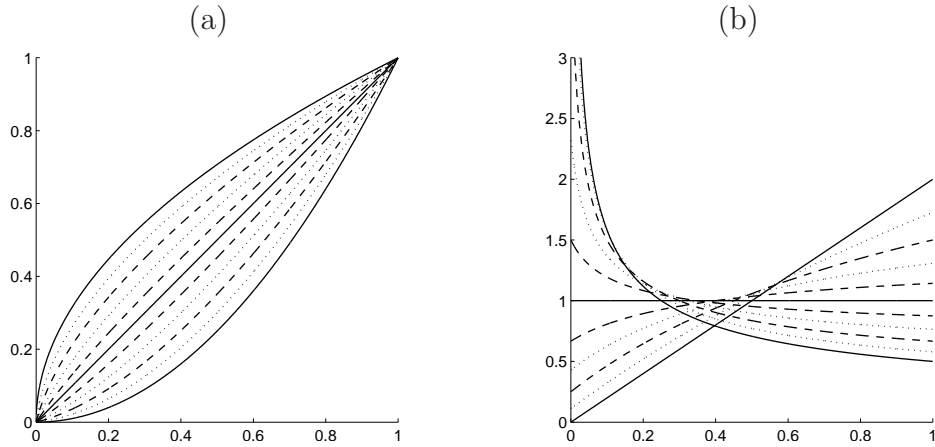


Figure 2: (a) $G_p(u; 1)$ and (b) $g_p(u; 1)$ plotted for $p = 0(0.1)1$, using different line-styles for clarity and linkage; in (a) and towards the left in (b), higher curves correspond to larger p .

ensues. And since $S_{F(\cdot; \ell)}(V) = \ell(5\ell^2 - 9)/540$, $E_{F(\cdot; \ell)}\{V^2 F(V; \ell)\} = (15 + 6\ell - \ell^2)/60$ and $E_{F(\cdot; \ell)}\{VF^2(V; \ell)\} = (105 + 21\ell - 7\ell^2 + \ell^3)/420$,

$$S_{G_p(\cdot; \ell)}(X) = (2p - 1) \ell \{63 - (41p^2 - 77p + 35) \ell^2\} / 3780.$$

Using the same building blocks, it is also found that

$$L_{2; G_p(\cdot; \ell)}(X) = 1/6 - (2p - 1)(p - 1) \ell^2 / 30$$

and

$$L_{3; G_p(\cdot; \ell)}(X) = (2p - 1) \ell \{7(3p^2 + 2) - 6(1 - p)^2 \ell^2\} / 420.$$

There were many tedious manipulations here, checked against special cases.

5. Additional remarks

5.1 On complementary distributions

Specifics of the complementary beta distribution are given in Jones (2002). Except for power law special cases, complementary beta and beta distributions differ, but it turns out that each complementary beta distribution is similar to a beta distribution with its parameters replaced by their reciprocals. Kumaraswamy distributions (Kumaraswamy, 1980, Jones, 2008), on the other hand, have c.d.f.s of the form $F_K(v; \alpha, \beta) = 1 - (1 - v^\alpha)^\beta$, $\alpha, \beta > 0$; complementary Kumaraswamy distributions therefore have c.d.f.s of the form

$$F_K^{-1}(v; \alpha, \beta) = \{1 - (1 - v)^{1/\beta}\}^{1/\alpha} = 1 - F_K(1 - v; 1/\beta, 1/\alpha);$$

if $V \sim F_K(\cdot; 1/\beta, 1/\alpha)$, then the complementary Kumaraswamy distribution is simply the distribution of $1-V$. (As Tahir & Nadarajah, 2015, point out, this distribution underlies the ‘exponentiated generalized distributions’ of Cordeiro, Ortega & da Cunha, 2013.)

5.2 On families of complementary distributions

This article has been concerned with understanding certain theoretical aspects of continuous distributions on $(0, 1)$; its purpose has not specifically been to add distributions of particular practical merit. For instance, with two parameters, one can expect to be able to control a distribution which includes both monotone and unimodal/anti-unimodal densities (e.g. the beta distribution) and it is unclear to what extent it is beneficial to employ a family like $G_p(\cdot; \ell)$ of Section 4 with more than one parameter yet monotone densities only. The latter family tractably gives rise to the further two-parameter family of two-piece distributions with densities

$$\gamma_p(u; \ell) = \begin{cases} g_p(2u; \ell), & 0 < u \leq 1/2, \\ g_p(2(1-u); \ell), & 1/2 < u < 1, \end{cases}$$

but these are limited to being symmetric about $u = 1/2$ and have a possibly unappealing cusp in their densities at $u = 1/2$.

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Appendix: proofs of results in Section 3

Using

$$E_H(T^2) = 1 - 2 \int_0^1 tH(t) dt \quad \text{and} \quad \int_0^1 tH^{-1}(t) dt = E_H\{TH(T)\},$$

gives rise to the following proof:

Proof of Theorem 3.5

$$\begin{aligned} E_{G_p}(X^2) &= 1 - 2 \int_0^1 xG_p(x)dx \\ &= 1 - 2 \left\{ (1-p)/3 + (2p-1) \int_0^1 xM_p^{-1}(x)dx \right\} / p \\ &= 1 - 2 \left[(1-p)/3 + (2p-1)E_{M_p}\{SM_p(S)\} \right] / p \\ &= 1 - 2 \left((1-p)/3 + (2p-1) \times \right. \\ &\quad \left. [p^2E_F\{VF(V)\} + p(1-p)\{1 + E_F(V^2)\}/2 + (1-p)^2/3] \right) / p \\ &= p(2p+1)/3 - 2p(2p-1)E_F\{VF(V)\} - (2p-1)(1-p)E_F(V^2). \end{aligned}$$

The theorem results by elementary methods.

Using

$$E_H(T^3) = 1 - 3 \int_0^1 t^2H(t) dt \quad \text{and} \quad \int_0^1 t^2H^{-1}(t) dt = E_H\{TH^2(T)\}$$

the proof of Theorem 3.6 ensues:

Proof of Theorem 3.6

$$\begin{aligned}
E_{G_p}(X^3) &= 1 - 3 \int_0^1 x^2 G_p(x) dx \\
&= 1 - 3 \left\{ (1-p)/4 + (2p-1) \int_0^1 x^2 M_p^{-1}(x) dx \right\} / p \\
&= 1 - 3 \left[(1-p)/4 + (2p-1) E_{M_p} \{ S M_p^2(S) \} \right] / p \\
&= 1 - 3 \left((1-p)/4 + (2p-1) \times \left[(1-p)(p^2 + 2p + 3)/12 \right. \right. \\
&\quad \left. \left. + p^3 E_F \{ V F^2(V) \} + p^2(1-p) E_F \{ V^2 F(V) \} \right. \right. \\
&\quad \left. \left. + p(1-p)^2 E_F(V^3)/3 \right] \right) / p \\
&= p(1+p+2p^2)/4 - (2p-1) \times \left[3p^2 E_F \{ V F^2(V) \} \right. \\
&\quad \left. + 3p(1-p) E_F \{ V^2 F(V) \} + (1-p)^2 E_F(V^3) \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_{G_p}(X) &= p(1+p+2p^2)/4 - (2p-1) \times \left[3p^2 E_F \{ V F^2(V) \} \right. \\
&\quad \left. + 3p(1-p) E_F \{ V^2 F(V) \} + (1-p)^2 E_F(V^3) \right] \\
&\quad - 3 \left\{ p - (2p-1) E_F(V) \right\} \times \left[p(2p+1)/3 \right. \\
&\quad \quad \left. - (2p-1) \left\{ 2p E \{ V F(V) \} + (1-p) E_F(V^2) \right\} \right] \\
&\quad + 2 \left[p^3 - 3p^2(2p-1) E_F(V) + 3p(2p-1)^2 \{ E_F(V) \}^2 \right. \\
&\quad \quad \left. - (2p-1)^3 \{ E_F(V) \}^3 \right]
\end{aligned}$$

which can be rearranged to give the result of the theorem.

Proof of Theorem 3.7

$$\begin{aligned}
L_{2;G_p} &= \int_0^1 \left[p(1-p)u - (1-p)^2 u^2 - 2(2p-1)(1-p)u M_p^{-1}(u) \right. \\
&\quad \left. + p(2p-1) M_p^{-1}(u) - (2p-1)^2 \{ M_p^{-1}(u) \}^2 \right] / p^2 du \\
&= \left[p(1-p)/2 - (1-p)^2/3 - 2(2p-1)(1-p) E_{M_p} \{ S M_p(S) \} \right. \\
&\quad \left. + p(2p-1) E_{M_p}(S) - (2p-1)^2 E_{M_p}(S^2) \right] / p^2 \\
&= \left(p(1-p)/2 - (1-p)^2/3 - 2(2p-1)(1-p) \times \right. \\
&\quad \left[p^2 E_F \{ V F(V) \} + p(1-p) \{ 1 + E_F(V^2) \} / 2 + (1-p)^2 / 3 \right] \\
&\quad \left. + p(2p-1) \{ p E_F(V) + 1 - p/2 \} \right. \\
&\quad \left. - (2p-1)^2 \{ p E_F(V^2) + (1-p)/3 \} \right) / p^2.
\end{aligned}$$

Further manipulation completes the proof.

Proof of Theorem 3.8

$$\begin{aligned}
L_{3;G_p} &= \int_0^1 [3p(1-p)^2u^2 - 2(1-p)^3u^3 - p^2(1-p)u \\
&\quad + 6p(2p-1)(1-p)uM_p^{-1}(u) - 6(1-p)^2(2p-1)u^2M_p^{-1}(u) \\
&\quad - 6(1-p)(2p-1)^2u\{M_p^{-1}(u)\}^2 + 3p(2p-1)^2\{M_p^{-1}(u)\}^2 \\
&\quad - 2(2p-1)^3\{M_p^{-1}(u)\}^3 - p^2(2p-1)M_p^{-1}(u)] / p^3 du \\
&= [p(1-p)^2 - (1-p)^3/2 - p^2(1-p)/2 \\
&\quad + 6p(2p-1)(1-p)E_{M_p}\{SM_p(S)\} - 6(1-p)^2(2p-1)E_{M_p}\{SM_p^2(S)\} \\
&\quad - 6(1-p)(2p-1)^2E_{M_p}\{S^2M_p(S)\} + 3p(2p-1)^2E_{M_p}(S^2) \\
&\quad - 2(2p-1)^3E_{M_p}(S^3) - p^2(2p-1)E_{M_p}(S)] / p^3 \\
&= ((1-p)(4p-4p^2-1)/2 + 6p(2p-1)(1-p) [p^2E_F\{VF(V)\} \\
&\quad + p(1-p)\{1+E_F(V^2)\}/2 + (1-p)^2/3] - 6(1-p)^2(2p-1) \times \\
&\quad [(1-p)(p^2+2p+3)/12 + p^3E_F\{VF^2(V)\} + p^2(1-p)E_F\{V^2F(V)\} \\
&\quad + p(1-p)^2E_F(V^3)/3] - 6(1-p)(2p-1)^2 [p^2E_F\{V^2F(V)\} \\
&\quad + p(1-p)\{1+2E_F(V^3)\}/3 + (1-p)^2/4] \\
&\quad + 3p(2p-1)^2 \{(1-p)/3 + pE_F(V^2)\} \\
&\quad - 2(2p-1)^3 \{(1-p)/4 + pE_F(V^3)\} \\
&\quad - p^2(2p-1) \{(1-p)/2 + pE_F(V)\}) / p^3 \\
&= [p^4(1-p)(1-2p)/2 + 6p^3(2p-1)(1-p)E_F\{VF(V)\} \\
&\quad + 3p^4(2p-1)E_F(V^2) - 6p^3(1-p)^2(2p-1)E_F\{VF^2(V)\} \\
&\quad - 6p^4(1-p)(2p-1)E_F\{V^2F(V)\} - 2p^5(2p-1)E_F(V^3) \\
&\quad - p^3(2p-1)E_F(V)] / p^3
\end{aligned}$$

and the result follows almost immediately.