

# Log-Location-Scale-Log-Concave Distributions for Survival and Reliability Analysis

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**Abstract.** We consider a novel sub-class of log-location-scale models for survival and reliability data formed by restricting the density of the underlying location-scale distribution to be log-concave. These models display a number of attractive properties. We particularly explore the shapes of the hazard functions of these, LLSLC, models. A relatively elegant, if partial, theory of hazard shape arises under a further minor constraint on the hazard function of the underlying log-concave distribution. Perhaps the most useful LLSLC models are contained in a class of three-parameter distributions which allow constant, increasing, decreasing, bathtub and upside-down bathtub shapes for their hazard functions.

*Key words:* bathtub; exponentiated Weibull; generalised  $F$ ; generalised gamma; hazard shape; log-concave; log-convex; mean residual life.

*Running head:* Log-Location-Scale-Log-Concave Distributions

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## 1. Introduction

This article concerns a novel class of models for nonnegative data which display a number of attractive properties. It comprises a slightly reduced subset of the well-known log-location-scale, or LLS, distributions discussed, for example, in Lawless (2003, especially Section 1.3.6 and Chapter 5) and Marshall & Olkin (2007, especially Chapter 12). This family of distributions and its properties are reviewed in Section 2. The reduction of the LLS distributions that we then make is to assume that the location-scale distributions which model the logarithms of the original data have log-concave densities, resulting in what we call log-location-scale-log-concave distributions, or LLSLC distributions for short. The additional properties that this endows are described in Section 3. Sections 4 and 5 explore the hazard functions of LLSLC distributions, with a relatively elegant, if partial, theory of hazard shape arising under the imposition of a further minor constraint on the hazard function of the underlying

log-concave distribution. Perhaps the most useful of these models are contained in the class of three-parameter distributions described in Section 5.1 which allow constant, increasing, decreasing, bathtub and upside-down bathtub shapes for their hazard functions. The article finishes, in Section 6, with a result concerning the mean residual life of LLSLC distributions.

## 2. Log-Location-Scale Distributions

Our starting point in this article is the class of location-scale distributions on  $\mathbb{R}$ , which have density and distribution functions of the respective forms

$$f_0(x; \mu, \sigma, \kappa) \equiv \frac{1}{\sigma} f\left(\frac{x - \mu}{\sigma}; \kappa\right), \quad F_0(x; \mu, \sigma, \kappa) \equiv F\left(\frac{x - \mu}{\sigma}; \kappa\right), \quad x \in \mathbb{R}.$$

Here,  $\mu \in \mathbb{R}$  is the location parameter,  $\sigma > 0$  is the scale parameter and any further, shape, parameters associated with this distribution are in  $\kappa$  (which may be a vector, but is typically one-dimensional in our work). Denote the random variable associated with  $f$  and  $F$  by  $X$ .

The LLS distributions are distributions on  $\mathbb{R}^+$  which arise via the log transformation  $X = \log(Y)$ ,  $Y = e^X$ . LLS distributions have density, distribution, survival, hazard and quantile functions, denoted  $g$ ,  $G$ ,  $\bar{G}$ ,  $h_G$  and  $Q_G \equiv G^{-1}$ , which are all immediately available in terms of the same functions of the location-scale distribution thus:

$$\begin{aligned} g(y; \theta, \lambda, \kappa) &= \frac{\lambda}{y} f\left\{\lambda \log\left(\frac{y}{\theta}\right); \kappa\right\}, & y > 0; \\ G(y; \theta, \lambda, \kappa) &= F\left\{\lambda \log\left(\frac{y}{\theta}\right); \kappa\right\}, & y > 0; \\ \bar{G}(y; \theta, \lambda, \kappa) &= \bar{F}\left\{\lambda \log\left(\frac{y}{\theta}\right); \kappa\right\}, & y > 0; \\ h_G(y; \theta, \lambda, \kappa) &= \frac{\lambda}{y} h_F\left\{\lambda \log\left(\frac{y}{\theta}\right); \kappa\right\}, & y > 0; \\ Q_G(u; \theta, \lambda, \kappa) &= \theta \exp\left\{\frac{1}{\lambda} Q_F(u; \kappa)\right\}, & 0 < u < 1. \end{aligned}$$

Here,  $\theta > 0$  and  $\lambda > 0$  are a reparametrisation of  $\mu, \sigma$  with attractive interpretation (Marshall & Olkin, 2007, pp. 228, 428). The location parameter  $\mu$  of  $f$  becomes, through  $\theta = e^\mu$ , the scale parameter of  $g$ . The scale parameter  $\sigma$  of  $f$  becomes, through  $\lambda = 1/\sigma$ , the power parameter of  $g$ , that is, the LLS class includes the distributions of  $Y_\lambda = Y_1^\lambda$  for all  $\lambda > 0$  whenever  $Y_1$  follows the LLS distribution with parameter  $\lambda = \sigma = 1$ . Log-location-scale

distributions might therefore be renamed scale-power distributions. We will often drop explicit dependence of functions on parameters for clarity in what follows.

Because of the convexity of the exponential transformation, an LLS distribution is always more skewed to the right in the classical van Zwet (1964) sense of convex transform order than the location-scale distribution from which it was transformed. In fact,  $\lambda$  is a skewness parameter for  $G$  in this sense, smaller  $\lambda$  corresponding to larger skewness.

The exponential transformation increases the weight in the right-hand tail of the LLS distribution relative to that of the original location-scale distribution: for instance, compare  $G(y) = F\{\lambda \log(y/\theta)\}$  with  $F(y)$  as  $y \rightarrow \infty$  (see also Section 3 below). Moments of LLS distributions can be written in terms of the moment generating function of  $f$  (Marshall & Olkin, 2007, p. 429):

$$E(Y^r) = \theta^r \int_{-\infty}^{\infty} \exp(rx/\lambda) f(x) dx.$$

It is easy to see that a pair of LLS distributions based on common  $f$  are stochastically ordered if their values of  $\lambda$  or of  $\theta$  are different but their values of the other parameters are the same.

Log-location-scale distributions also have Fisher information matrices with an attractively simple structure; this will not be considered here.

### 3. Log-Location-Scale-Log-Concave Distributions

**Definition.** LLSLC distributions are those LLS distributions based on choices for  $f$  — and hence  $f_0$  — that are log-concave i.e.  $(\log f)''(x) \leq 0$  for all  $x \in \mathbb{R}$ .

Distributions with log-concave densities have a number of interesting properties (An, 1995, 1998, Bagnoli & Bergstrom, 2005, Marshall & Olkin, 2007, Section 4.B, Walther, 2009). However, focus here is on the resulting properties of  $g$ , obtained after exponential transformation. These properties are, of course, in addition to all the properties outlined in Section 2 which continue to apply to LLSLC distributions.

A first result is that log-concavity of  $f$  implies unimodality of  $g$ , where ‘unimodality’ allows the mode to be at 0 (so that  $g$  is then monotone decreasing). To see this, leaving out the dependence on parameters except where necessary,  $(\log g)'(y) = \mathcal{G}(y)/y$  where

$$\mathcal{G}(y) = \lambda(\log f)' \left\{ \lambda \log \left( \frac{y}{\theta} \right) \right\} - 1.$$

The log-concavity of  $f$  implies that  $\mathcal{G}'(y) < 0$ , for all  $y > 0$ . Since the log-concavity of  $f$  implies that  $f$  is unimodal on  $\mathbb{R}$  (Marshall & Olkin, 2007, Proposition 4.B.2),  $(\log f)'(x) > 0$  for  $x < x_0$  where  $x_0$  is the mode of  $f$ , and is negative thereafter; therefore,  $\mathcal{G}(y)$ , and hence  $(\log g)'(y)$ , is either positive or negative for small  $y$  and is negative for large enough  $y$ . In fact, the mode of  $g$  will be at some  $y_0 < \theta \exp(x_0/\lambda)$ .

It is certainly not the case that  $g$  is itself log-concave, nor would one desire this. This is because log-concave distributions have light-to-moderate tails. In fact, the heaviest possible tails of  $f$  are simple exponential (An, 1995, 1998), in the sense that  $f(x) \sim \exp(\xi x)$  for some  $\xi > 0$  as  $x \rightarrow -\infty$  and/or  $f(x) \sim \exp(-\eta x)$  for some  $\eta > 0$  as  $x \rightarrow \infty$  (examples include Laplace and logistic distributions). It is easy to see, however, that the tailweight-increasing property of the exponential transformation allows  $g$  to have a heavy tail: for example, if  $f$  has an exponential right-tail then  $g(y) \sim y^{-(\eta\lambda+1)}$  as  $y \rightarrow \infty$ , has a power, or Pareto, tail with tail index  $\eta\lambda$ . Else, one or both tails of  $f$  are lighter than exponential, which we will refer to as ‘super-exponential’, so if the right-tail of  $f$  is super-exponential, the tail of  $g$  is lighter than power-tailed (for example, log-normal tail for  $g$  from normal tail for  $f$ , Weibull tail for  $g$  from extreme-value tail for  $f$ ). For more on this, see Section 4 below on the properties of  $h_G$ .

Convolutions of log-concave distributions are again log-concave distributions (e.g. Marshall & Olkin, Proposition 4.B.3). This translates to saying that the distribution of the product of a pair of independent LLSLC random variables is also distributed according to an LLSLC distribution.

Location-log-concave distributions have monotone likelihood ratio in  $\mu$  (e.g. Marshall & Olkin, 2007, p.59). This translates immediately to log-location-log-concave distributions (LLSLC distributions with fixed  $\lambda$  and  $\kappa$ ) having monotone likelihood ratio in  $\theta$ . Also, because  $f$  is log-concave,  $\bar{F}$  is log-concave, and so  $h_F(x) = -(\log \bar{F})'(x)$  is increasing (e.g. Marshall & Olkin, 2007, Proposition 4.B.8.a). This implies the hazard rate ordering of LLSLC distributions in  $\theta$  when the other parameters are fixed.

#### 4. Hazard Functions of LLSLC Distributions

Using the relationship between the hazard function of an LLSLC distribution,  $h_G(y)$ , and that of its underlying log-concave distribution,  $h_F(x)$ , the number of modes of  $h_G$  will be the same as the number of modes of

$$\log h_G(y) = \log \lambda - \log y + \log h_F \left\{ \lambda \log \left( \frac{y}{\theta} \right) \right\}$$

over  $y > 0$  and thence, setting  $x = \lambda \log(y/\theta)$ , of

$$t(x) \equiv \log(\lambda/\theta) - \frac{x}{\lambda} + \log h_F(x)$$

over  $x \in \mathbb{R}$ . Now,

$$t'(x) = -\frac{1}{\lambda} + (\log h_F)'(x).$$

Thus, since  $f$  is log-concave,  $\log h_F$  is increasing, and so the derivative of  $t$  consists of a positive function plus a negative constant.

An elegant theory of LLSLC hazard shape arises for a constrained subset of LLSLC distributions; the constraint appears not to be a very restrictive one.

**Constraint.** The constrained LLSLC (CLLSLC) distributions of interest are those LLSLC distributions based on choices for  $f$  such that its hazard function  $h_F$  is log-concave, log-convex or both. We abbreviate this requirement to  $h_F$  being ‘log-concavex’.

For CLLSLC distributions, there are three cases to consider:

**Case 1:**  $(\log h_F)''(x) = 0$ , so that  $\log h_F(x) = \delta + \beta x$ , for some  $\delta \in \mathbb{R}$  and  $\beta > 0$ . Then,  $t(x) = (\beta - 1/\lambda)x + \text{constant}$ . The corresponding  $h_G$ 's are monotone; in fact, they are power functions, and hence  $g$  is the Weibull distribution: from the definition,

$$h_G(y) = \frac{\lambda e^\delta}{\theta^{\beta\lambda}} y^{\beta\lambda - 1}$$

which is increasing (constant) decreasing as  $\beta\lambda > (=) < 1$ .

**Case 2:**  $(\log h_F)''(x) < 0$ . In this case,  $t'$  is monotone decreasing. So  $t$ , and hence  $h_G$ , can be increasing, unimodal or decreasing, but it cannot be bathtub-shaped. Here and for the rest of the paper, in the context of hazard functions, we use ‘unimodal’ to mean ‘upside-down bathtub shaped’.

**Case 3:**  $(\log h_F)''(x) > 0$ . In this case,  $t'$  is monotone increasing. So  $t$ , and hence  $h_G$ , is increasing, bathtub or decreasing, but not unimodal.

Taken together, CLLSLC distributions can only be monotone increasing or decreasing (including constant), bathtub (by which we mean decreasing then increasing) or unimodal (by which we mean increasing then decreasing). That is, the derivative of the hazard function can have at most one change of sign.

To match hazard shapes with parameter values, it is helpful to consider the tail behaviour of  $f$ . This involves four cases which are summarised in Table 1 below.

**Case I:** exponential left-hand tail. As  $x \rightarrow -\infty$ ,  $h_F(x) \sim f(x) \sim \gamma_L e^{\xi x}$ ,  $\gamma_L, \xi > 0$ , say, and so  $t(x) \sim (\xi - 1/\lambda)x + \text{constant}$ . This goes to  $-\infty$  (constant)  $\infty$  as  $x \rightarrow -\infty$ , as  $\xi > (=) < 1/\lambda$ .

**Case II:** exponential right-hand tail. As  $x \rightarrow \infty$ ,  $f(x) \sim \gamma_R e^{-\eta x}$ ,  $\gamma_R, \eta > 0$ ,  $\overline{F}(x) \sim (\gamma_R/\eta)e^{-\eta x}$ ,  $h_F(x) \sim \eta > 0$ , and so  $t(x) \sim -\infty$ .

**Case III:** super-exponential left-hand tail. As  $x \rightarrow -\infty$ , write  $f(x) \sim e^{\ell(x)}$  for any  $\ell(x)$  tending to minus infinity faster than  $x$ . Then,  $h_F(x) \sim f(x) \sim e^{\ell(x)}$  so that  $t(x) \sim \log(\lambda/\theta) + \log\{e^{-x/\lambda + \ell(x)}\}$  tends to  $-\infty$ .

**Case IV:** super-exponential right-hand tail. As  $x \rightarrow \infty$ , write  $f(x) \sim e^{-r(x)}$  for any  $r(x)$  tending to infinity faster than  $x$ . Then, using l'Hôpital's rule,  $h_F(x) \sim r'(x)$  so that  $t(x) \sim \log(\lambda/\theta) + \log\{r'(x)e^{-x/\lambda}\}$ . It follows that, as  $x \rightarrow \infty$ ,  $t(x)$  can tend to any of  $-\infty$ , constant or  $\infty$ ; example choices of  $r$  for each case are:  $r(x) = x^\gamma$  ( $\gamma > 1$ ),  $r(x) = e^{x/\lambda}$  and  $r(x) = e^{x^\beta}$  ( $\beta > 0$ ), respectively.

Table 1. Limits of  $t$ .

Tail of $f$	Left	Right
Exponential	any $(-\infty, \text{const}, \infty)$	$-\infty$
Super-exponential	$-\infty$	any $(-\infty, \text{const}, \infty)$

The consequences for shapes of  $h_G$  are immediate and summarised in Table 2:

Table 2. Shapes of  $h_G$  and their dependence on tails of  $f$ .

Tails of $f$	Right: exponential	Right: super-exponential
Left: exponential	decreasing or unimodal	increasing or constant or decreasing or bathtub or unimodal
Left: super-exponential	unimodal	increasing or unimodal

## 5. Special Cases of LLSLC Distributions

### 5.1 LLSLC Distributions with Increasing or Constant or Decreasing or Bathtub or Unimodal Hazard Functions

The most interesting subset of CLLSLC distributions would appear to be those occupying the top right-hand cell of Table 2. These are distributions with three parameters (when  $\kappa$  in  $f$  is scalar), previously referred to as scale ( $\theta > 0$ ), power ( $\lambda > 0$ ) and shape ( $\kappa$ ) parameters, which parsimoniously afford this

wide and attractive variety of shapes. Since  $f$  must have an exponential left-hand tail, parametrise it so that the exponential rate of decay is  $\kappa > 0$  (that is,  $\beta = \kappa$  in Case 1,  $\xi = \kappa$  in Case I). With a reparametrisation of the form  $\alpha = \kappa\lambda > 0$ , a particularly attractive set of examples of such distributions has hazard functions with the properties:

- (i) as  $y \rightarrow 0$ ,  $h_G(y) \sim y^{\alpha-1}$ ;
- (ii) as  $y \rightarrow \infty$ ,  $h_G(y) \sim y^{\lambda-1}$ .

Notice that as  $y \rightarrow 0$ ,  $h_G$  is zero (nonzero constant) infinity as  $\alpha > (=) < 1$  and that as  $y \rightarrow \infty$ ,  $h_G$  is zero (nonzero constant) infinity as  $\lambda < (=) > 1$ . The parameters of these distributions can therefore be reinterpreted as controlling scale ( $\theta$ ), hazard behaviour near zero ( $\alpha$ ) and hazard behaviour for large  $y$  ( $\lambda$ ).

*Table 3. Shapes of  $h_G$  for LLSLC distributions of the type described in this subsection when neither  $\alpha$  nor  $\lambda$  equals 1.*

	$\alpha < 1$	$\alpha > 1$
$\lambda < 1$	decreasing	unimodal
$\lambda > 1$	bathtub	increasing

Moreover, for CLLSLC distributions with such limiting hazard behaviour, ‘joining the tail’ considerations immediately give the overall shape of the hazard function to be as in Table 3, for all combinations of values of  $\alpha$  and  $\lambda$  with neither of them taking the value 1. Behaviour in the remaining ‘threshold’ cases where either or both of  $\alpha$  and  $\lambda$  equal 1 can be dealt with on a case-by-case basis.

We know of three examples of such distributions (each presented with their scale parameter  $\theta$  suppressed):

**Example 1:** the generalized gamma (GG) distribution (Stacy, 1962, Cox et al., 2007). In the current parametrisation, this has density

$$g_{GG}(y) = \lambda y^{\alpha-1} \exp(-y^\lambda) / \Gamma(\alpha/\lambda)$$

and hazard

$$h_{G;GG}(y) = \frac{\lambda y^{\alpha-1} \exp(-y^\lambda)}{\Gamma(\alpha/\lambda) - \Gamma(y^\lambda; \alpha/\lambda)}$$

where  $\Gamma(z; \delta) = \int_0^z w^{\delta-1} e^{-w} dw$  is the incomplete gamma function. It includes the Weibull and gamma (and hence exponential) distributions, as well as the generalized (or power of) half-normal distribution (Cooray & Ananda, 2008),

as special cases and has the lognormal distribution as a limiting case. The GG distribution is an LLSLC distribution because

$$f_{GG}(x) = \exp(\kappa x - e^x)/\Gamma(\kappa)$$

is log-concave. It is shown in Appendix A that  $h_{F;GG}$  is log-concave for  $\kappa > 1$  and log-convex for  $\kappa < 1$ . That the behaviour of the GG hazard function follows Table 3 is confirmed in the seminal paper of Glaser (1980) where it provides an example of how shapes of hazard functions  $h_G$  can be implied by shapes of the function  $-g'/g$ . Indeed, those considerations cover the threshold cases too, and so the GG hazard function behaves as in Table 4. See also Cox et al. (2007) and Cox & Matheson (2014) where it is argued that this makes the GG distribution of particular value in survival analysis. A downside, perhaps, which is also relevant to dealing with censored data, is the appearance of the incomplete gamma function in its survival and hazard functions.

Table 4. Shapes of  $h_{G;GG}$ ,  $h_{G;EW}$  and  $h_{G;PGW}$ .

	$\alpha < 1$	$\alpha = 1$	$\alpha > 1$
$\lambda < 1$	decreasing	decreasing	unimodal
$\lambda = 1$	decreasing	constant	increasing
$\lambda > 1$	bathtub	increasing	increasing

**Example 2:** the exponentiated Weibull (EW) distribution (Mudholkar & Srivastava, 1993, Mudholkar et al., 1995). In the current parametrisation,

$$g_{EW}(y) = \alpha\{1 - \exp(-y^\lambda)\}^{(\alpha/\lambda)-1}y^{\lambda-1}\exp(-y^\lambda)$$

and

$$h_{G;EW}(y) = \frac{\alpha\{1 - \exp(-y^\lambda)\}^{(\alpha/\lambda)-1}y^{\lambda-1}\exp(-y^\lambda)}{1 - \{1 - \exp(-y^\lambda)\}^{\alpha/\lambda}}.$$

Note the explicit distribution function  $G_{EW}(y) = \{1 - \exp(-y^\lambda)\}^{\alpha/\lambda}$  (but neither the hazard function nor its left-hand limit are as erroneously claimed in the review paper of Nadarajah et al., 2013, p.841). The EW distribution is an LLSLC distribution because

$$f_{EW}(x) = \kappa\{1 - \exp(-e^x)\}^{\kappa-1}\exp(x - e^x)$$

can be shown (see Appendix B) to be a log-concave density. We strongly conjecture that  $h_{F;EW}$  is, like  $h_{F;GG}$ , log-concave for  $\kappa > 1$  and log-convex for  $\kappa < 1$ , and will proceed assuming this to be the case. See Figure 2 in Appendix



C for graphical evidence; extensive numerical computation also supports our claim yet, despite strenuous efforts, we have been unable to prove it. That the EW hazard function also follows Table 4 is confirmed in Section 2 of Mudholkar et al. (1995). The EW distribution includes the Weibull and exponentiated exponential (and hence exponential) distributions, as well as the Burr Type X distributions, as special cases. The similarity of the exponentiated Weibull distribution to the generalized gamma distribution has recently been explored in detail by Cox & Matheson (2014). The similarity of the exponentiated exponential distribution to the gamma distribution was earlier investigated by Gupta & Kundu (2003).

**Example 3:** the power generalized Weibull (PGW) distribution (Bagdonavičius & Nikulin, 2002, Nikulin & Haghighi, 2006, 2009, apparently independently treated by Dimitrakopoulou et al., 2007). Probably the simplest direct construct of a tractable hazard function with limiting properties (i) and (ii) is

$$h_{G;PGW}(y) = \lambda y^{\alpha-1} (1 + y^\alpha)^{(\lambda/\alpha)-1};$$

this is the hazard function of Nikulin and colleagues' PGW distribution, which has survival function

$$\bar{G}_{PGW}(y) = \exp\{1 - (1 + y^\alpha)^{\lambda/\alpha}\}$$

and density

$$g_{PGW}(y) = \lambda y^{\alpha-1} (1 + y^\alpha)^{(\lambda/\alpha)-1} \exp\{1 - (1 + y^\alpha)^{\lambda/\alpha}\}.$$

The PGW distribution reduces to the Weibull distribution when  $\alpha = \lambda$ . The appropriate scaled log of a PGW random variable has density

$$f_{PGW}(x) = e^{\kappa x} (1 + e^{\kappa x})^{(1/\kappa)-1} \exp\{1 - (1 + e^{\kappa x})^{1/\kappa}\}.$$

Straightforward manipulations briefly given in Appendix D confirm that  $\log f_{PGW}$  is concave and that  $\log h_{F;PGW}$  is convex (linear) concave as  $\kappa < (=) > 1$ . In fact, the derivative of  $\log h_{G;PGW}(y)$  with respect to  $y$  is a positive function of  $y$  times  $\alpha - 1 + (\lambda - 1)y^\alpha$ , from which the hazard shapes in Table 4 immediately follow.

A graphical comparison of the hazard functions of the GG, EW and PGW distributions is given in Figure 1. Each frame provides representatives of one of the four possible nonconstant hazard regimes. For each distribution, the values of  $\alpha$  and  $\lambda$  are the same, but scalings are changed to equate the

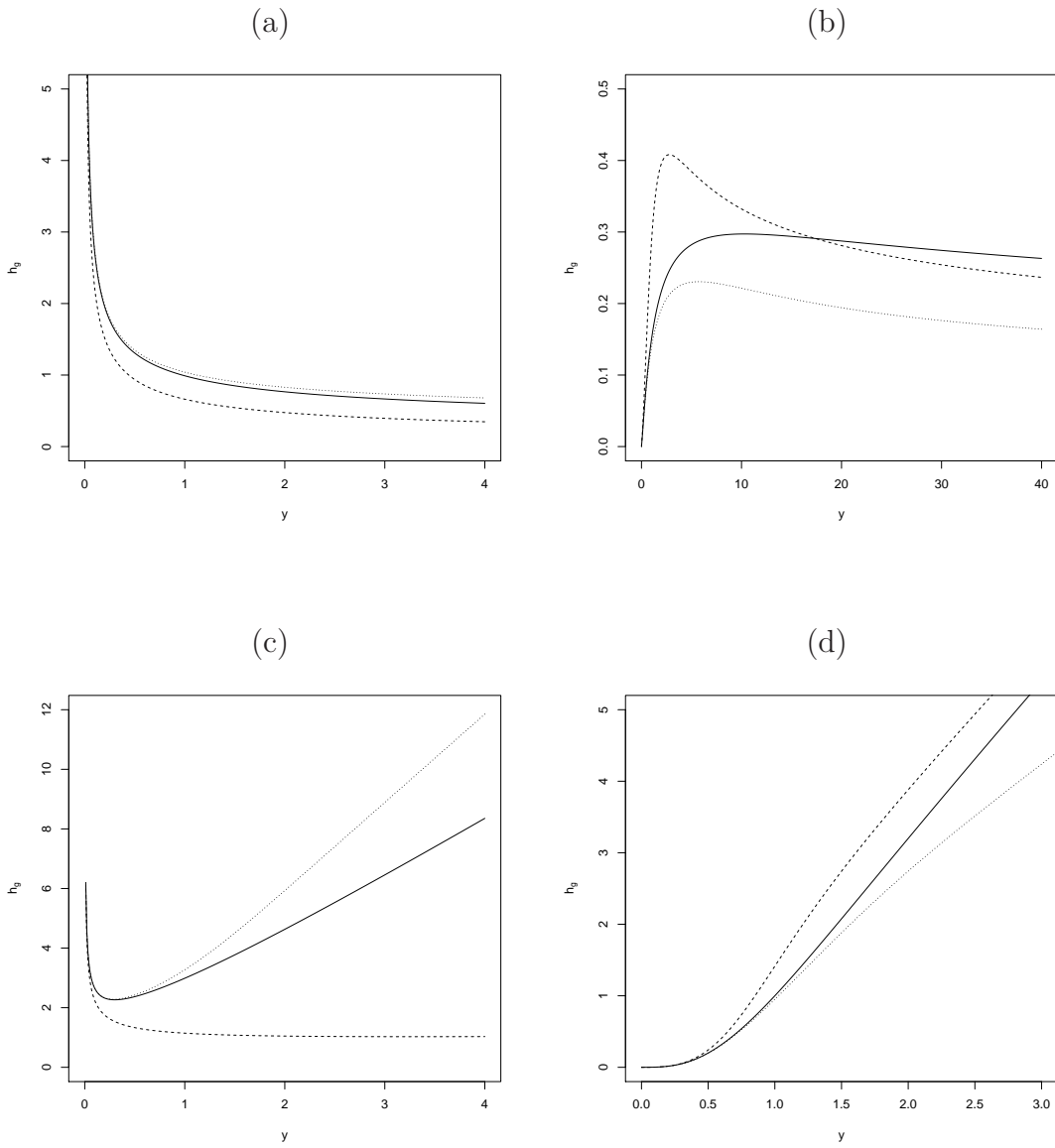


Figure 1: Examples of hazard functions of GG (solid), EW (dotted) and PGW (dashed) distributions, with (a)  $\lambda = \frac{3}{4}, \alpha = \frac{3}{8}$ , (b)  $\lambda = \frac{3}{4}, \alpha = \frac{9}{4}$ , (c)  $\lambda = 2, \alpha = \frac{1}{2}$ , (d)  $\lambda = 2, \alpha = 4$ .

constant multiples of the hazard as  $y \rightarrow 0$ . That is, we plot  $\theta_K^{-1}h_{G;K}(\theta_K^{-1}y)$ ,  $K = GG, EW, PGW$ , where  $\theta_{GG} = 1$ ,  $\theta_{EW} = \{\Gamma((\alpha/\lambda) + 1)\}^{1/\alpha}$  and  $\theta_{PGW} = \{\Gamma(\alpha/\lambda)\}^{1/\alpha}$ . The hazard functions of all three distributions are generally similar in monotone cases, but differ more in unimodal and bathtub cases. In particular, while the GG and EW hazard functions remain broadly similar, the PGW hazard differs rather more from them; the PGW hazard seems to have a sharper mode than the others in unimodal cases and a shallower antimode than the others in bathtub cases. At the least, while the PGW distribution shares the attractive set of hazard shapes of the GG and EW distributions (Cox & Matheson, 2014), with the same number of parameters, it seems to differ more from its GG and EW competitors than those two do from each other.

### 5.2 LLSLC Distributions with Decreasing or Unimodal Hazard Functions

LLSLC distributions in the top left-hand cell of Table 2 include the most heavy-tailed ones; they are treated more briefly here as they might not have such importance for survival and reliability data as they do in some other contexts (e.g. financial). These distributions are based on  $f$ 's with two exponential tails; a wide variety of such distributions is covered by the family of distributions in Jones (2008). Members of this family have densities of the form

$$f(x) \propto \exp\{\kappa x - (\kappa + \tau)W(x)\},$$

$\kappa, \tau > 0$ . Here,  $W(x)$  is the first iterated distribution function of a symmetric distribution on  $\mathbb{R}$ , itself with tails that are lighter than Cauchy, that is,  $W'(x)$  is the distribution function of a not-extremely-heavy-tailed symmetric distribution. Such  $W$  satisfy  $W(x) - W(-x) = x$  and are such that  $W(x) \rightarrow 0$  as  $x \rightarrow -\infty$ ,  $W(x) \sim x$  as  $x \rightarrow \infty$ . These densities are log-concave because  $(\log f)''(x) = -(\kappa + \tau)W''(x)$  and  $W''(x) > 0$  is a (symmetric) density function. Unfortunately, these distributions are not very tractable in general.

Nonetheless, a prime example of such an  $f$  is the log  $F$  distribution arising from  $W(x) = \log(1 + e^x)$  (the iterated distribution function of the logistic distribution); it has  $f_{GF}(x) \propto e^{\kappa x} / (1 + e^x)^{\kappa + \tau}$ . It follows that

$$g_{GF}(y) \propto y^{\alpha-1} / (1 + y^\lambda)^{(\alpha/\lambda) + \tau}.$$

This is the generalized  $F$ , generalized beta of the second kind, or Feller-Pareto, distribution (e.g. Johnson et al., 1994, Chapter 27, Cox, 2008, Arnold, 2014). The generalized  $F$  distribution does not have decreasing or unimodal hazards for all choices of its parameters (Cox, 2008).

**Special Case 1:**  $\alpha = \lambda$  ( $\kappa = 1$ ), the Burr Type XII, or Pareto Type IV or Singh-Maddala, distribution. This is very tractable with, inter alia,  $h_{G;BXII}(y) = \lambda\tau y^{\lambda-1}/(1+y^\lambda)$ . It is easy to show directly that  $\log h_{F;BXII}$  is concave and that  $h_{G;BXII}$  is decreasing for  $\lambda \leq 1$  and unimodal otherwise. Note that  $\tau$  acts only as a proportionality coefficient in this hazard function.

The result of applying our approach to the generalized  $F$  distribution is then easy to state:  $\log h_{F;GF}$  is concave if and only if  $\kappa \geq 1$ , and so  $h_{G;GF}$ , which tends to zero as  $x \rightarrow \infty$  for all values of its parameters, is guaranteed to be either decreasing or unimodal when  $\kappa \geq 1$  (equivalently, when  $\alpha \geq \lambda$ ). This is proved in Appendix E.

This ‘high level’ information — which takes no direct account of  $\lambda$  — can be complemented with more detailed information about when  $h_{G;GF}$  is decreasing or unimodal given by Glaser’s method (Marshall & Olkin, 2007, Proposition C.4(1),(ii)). Briefly, the hazard rate is unimodal if  $\lambda\kappa > 1$  or if  $\kappa = 1/\lambda < 1$ . It is decreasing if  $\lambda\kappa < 1$  and either  $\lambda\{\kappa(1+\lambda) + \tau(\lambda-1)\} \leq 2$  or  $\kappa(1+\lambda)^2 + \tau(1-\lambda)^2 < 4$ , or if  $\kappa = 1/\lambda \geq 1$ .

**Special Case 2:**  $\lambda = 1$ , the (scaled)  $F$ , or beta of the second kind, distribution. The  $F$  hazard function is decreasing for  $\alpha = \kappa \leq 1$  and unimodal otherwise.

### 5.3 LLSLC Distributions with Increasing or Unimodal or Just Unimodal Hazard Functions

The lower two cells in Table 2 are perhaps least interesting and will be considered even more briefly. Distributions in the bottom left-hand cell, with unimodal hazards only, arise from distributions in the top right-hand cell by replacing  $f$  by the distribution of  $-X$ , density  $f(-x)$ . The corresponding  $g$  distributions are the distributions of  $1/Y$  where  $Y$  follows any distribution in the top right-hand cell of Table 2.

Distributions in the bottom right-hand cell of Table 2 can be constructed in bespoke fashion from densities  $f$  with a pair of light, superexponential, tails.

## 6. Mean Residual Life of LLSLC Distributions

Confine attention in this section to LLSLC distributions for which the mean  $\mu_G$  exists. The mean residual life function is, of course,

$$M_G(y) \equiv E_G(Y - y | Y \geq y) = \frac{\int_y^\infty \overline{G}(t) dt}{\overline{G}(y)} = \frac{\mathcal{I}_G(y)}{\overline{G}(y)},$$

say. Note that  $M_G(0) = \mu_G$ .

Now, for LLSLC  $G$ , we can write

$$\mathcal{I}_G(y) = \int_y^\infty \overline{F}(\lambda \log(t/\theta)) dt = \frac{\theta}{\lambda} \int_{\lambda \log(t/\theta)}^\infty e^{w/\lambda} \overline{F}(w) dw.$$

Since  $F$  is log-concave,  $h_F$  is increasing, and we have

$$\begin{aligned} h_F(\lambda \log(y/\theta)) \mathcal{I}_G(y) &\leq \int_y^\infty h_F(\lambda \log(t/\theta)) \overline{F}(\lambda \log(t/\theta)) dt \\ &= \int_y^\infty f(\lambda \log(t/\theta)) dt = \frac{\theta}{\lambda} \int_{\lambda \log(y/\theta)}^\infty e^{w/\lambda} f(w) dw \\ &= \frac{1}{\lambda} \{y \overline{F}(\lambda \log(y/\theta)) + \mathcal{I}_G(y)\}, \end{aligned}$$

using integration by parts. (The fact that the term  $e^{w/\lambda} \overline{F}(w) \rightarrow 0$  as  $w \rightarrow \infty$  follows from the conditions for  $\mu_G$  to exist, namely that  $f$  have either a super-exponential tail, or an exponential tail with  $\eta > 1/\lambda$ , as  $x \rightarrow \infty$ .) Rewriting the above inequality in terms of  $G$  and multiplying throughout by  $\lambda$ , we have

$$y h_G(y) \mathcal{I}_G(y) \leq y \overline{G}(y) + \mathcal{I}_G(y)$$

which can be rearranged to yield  $(y h_G(y) - 1) M_G(y) \leq y$ . When  $h_G(y) \leq 1/y$  this is unhelpful, but when  $h_G(y) > 1/y$  we have the bound

$$M_G(y) \leq \frac{1}{h_G(y) - (1/y)}.$$

The bound is particularly relevant for large  $y$ , where it adds to the general property of mean residual life functions that  $M_G(y) \sim 1/h_G(y)$  as  $y \rightarrow \infty$ .

## 7. Conclusions

This paper has presented a unified view of distributions for survival and reliability data which are not only log-location-scale distributions, with the advantages thereof, but a subset of them which arise from log-concave distributions on the ‘logged’ scale. These LLSLC distributions are additionally necessarily unimodal and closed under multiplication of random variables.

A particular focus has been on the important question of shapes of hazard functions. These shapes can be well understood within the LLSLC framework (Sections 4 and 5) allowing the categorisation of certain existing distributions

and the potential for constructing other distributions with desirable hazard structures. Perhaps the most useful LLSLC models are those of Section 5.1, three-parameter distributions allowing constant, increasing, decreasing, bathtub and unimodal hazard functions. Our work sheds further theoretical light on the strong similarities — observed by Cox & Matheson (2014) — between two especially useful, and highly recommended, distributions of this type, the generalised gamma and exponentiated Weibull distributions. Superficially, we expected to agree with Cox & Matheson that “An advantage of the EW family is that it is easier to work with than the GG”; paradoxically, we were able to prove our conjecture of the logconcavity of  $h_F$  only for the GG distribution and not for the EW!

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### Appendix A: proof that $h_{F;GG}$ is log-concave

Define

$$G_{\kappa}(x) = \Gamma(\kappa) - \Gamma(e^x; \kappa) = \int_{e^x}^{\infty} y^{\kappa-1} e^{-y} dy,$$

so that

$$\log h_{F;GG}(x) = \kappa x - e^x - \log G_{\kappa}(x),$$

and define  $L(x) = \exp\{(\kappa - 1)x - e^x\}$ . Note that

$$G'_\kappa(x) = -e^x L(x) \quad \text{and} \quad G''_\kappa(x) = (e^x - \kappa)e^x L(x)$$

and, by integration-by-parts,

$$G_{\kappa+1}(x) = e^x L(x) + \kappa G_\kappa(x) \quad \text{and} \quad G_\kappa(x) = L(x) + (\kappa - 1)G_{\kappa-1}(x).$$

Now,

$$(\log h_{F;GG})''(x) = -e^x - (\log G_\kappa)''(x) = \frac{e^x}{G_\kappa^2(x)} \mathcal{A}_\kappa(x)$$

where

$$\begin{aligned} \mathcal{A}_\kappa(x) &= e^x L^2(x) + (\kappa - e^x)L(x)G_\kappa(x) - G_\kappa^2(x) \\ &= L(x)\{e^x L(x) + \kappa G_\kappa(x)\} - G_\kappa(x)\{e^x L(x) + G_\kappa(x)\} \\ &= L(x)G_{\kappa+1}(x) - G_\kappa(x)\{e^x L(x) + \kappa G_\kappa(x)\} + (\kappa - 1)G_\kappa^2(x) \\ &= \{L(x) - G_\kappa(x)\}G_{\kappa+1}(x) + (\kappa - 1)G_\kappa^2(x) \\ &= (\kappa - 1)\{G_\kappa^2(x) - G_{\kappa+1}(x)G_{\kappa-1}(x)\}. \end{aligned}$$

The term in curly brackets is negative by the Cauchy-Schwartz inequality, so  $h_{F;GG}$  is log-convex if  $\kappa < 1$  and is log-concave if  $\kappa > 1$ , as required.

## Appendix B: proof that $f_{EW}$ is log-concave

$$\begin{aligned} (\log f_{EW})''(x) &= -e^x + (\kappa - 1) \frac{e^x \exp(-e^x) \{1 - e^x - \exp(-e^x)\}}{\{1 - \exp(-e^x)\}^2} \\ &= -\frac{e^x}{\{1 - \exp(-e^x)\}^2} \mathcal{N}_\kappa(x) \end{aligned}$$

where

$$\mathcal{N}_\kappa(x) = 1 - (\kappa + 1) \exp(-e^x) + \kappa \exp(-2e^x) + (\kappa - 1)e^x \exp(-e^x)$$

Therefore,  $f_{EW}$  will be log-concave if  $\mathcal{N}_\kappa(x) > 0$  or equivalently if  $\mathcal{N}_\kappa(y) > 0$  where  $y = e^x > 0$  and

$$\mathcal{N}_\kappa(y) = \{1 - (1 + y)e^{-y}\} + \kappa e^{-y} \{e^{-y} + y - 1\}$$



It is, in fact, true that  $\mathcal{N}_\kappa(y) > 0$ ; this is because  $\kappa > 0$ ,  $e^{-y} + y - 1 > 0$  and  $1 - (1 + y)e^{-y} > 0$ . The latter two inequalities arise from the standard inequalities for the exponential function

$$\frac{z}{1+z} < 1 - e^{-z} < z, \quad z > -1,$$

the first immediately, the second after writing  $1 - (1 + y)e^{-y} = (1 + y)(1 - e^{-y}) - y$ .

### Appendix C: graphical evidence that $h_{F;EW}$ is log-concave

Plots of the second derivative of  $\log h_{F;EW}(x)$  suggestive of its being positive for all  $\kappa < 1$  and negative for all  $\kappa > 1$  (it is zero for  $\kappa = 0$ ) are given in Figure 2.

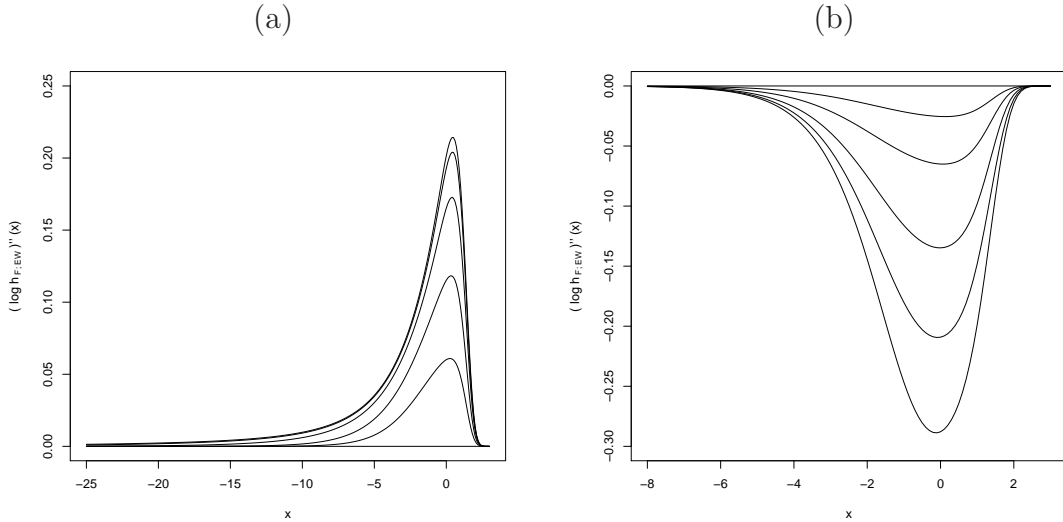


Figure 2: Plots of  $(\log h_{F;EW})''(x)$ : (a) in order of decreasing maximum, for  $\kappa = 0.05, 0.1, 0.25, 0.5, 0.75, 1$ ; (b) in order of increasing minimum, for  $\kappa = 2, 1.75, 1.5, 1.25, 1.1, 1$ .

### Appendix D: proof that $f_{PGW}$ is log-concave and $h_{F;PGW}$ is log-concave

$$(\log h_{F;PGW})''(x) = (1 - \kappa)\kappa \frac{e^{\kappa x}}{(1 + e^{\kappa x})^2}$$

and the claimed log-concavity of  $h_{f:PGW}$  is clear. In addition,

$$\begin{aligned} (\log f_{PGW})''(x) &= (\log h_{F:PGW})''(x) + (\log \overline{F}_{PGW})''(x) \\ &= (1 - \kappa)\kappa \frac{e^{\kappa x}}{(1 + e^{\kappa x})^2} - e^{\kappa x} (1 + e^{\kappa x})^{(1/\kappa)-2} (\kappa + e^{\kappa x}) \end{aligned}$$

and the log-concavity of  $f_{PGW}$  when  $\kappa \geq 1$  is immediate. For  $\kappa < 1$ , note that

$$(\log f_{PGW})''(x) = \frac{e^{\kappa x}}{(1 + e^{\kappa x})^2} \mathcal{O}_\kappa(x)$$

where

$$\mathcal{O}_\kappa(x) = (1 - \kappa)\kappa - (1 + e^{\kappa x})^{1/\kappa} (\kappa + e^{\kappa x}).$$

However,  $\mathcal{O}_\kappa(x)$  is easily seen to be a decreasing function of  $x$  and so is maximised when  $x \rightarrow -\infty$ , taking a maximised value of  $(1 - \kappa)\kappa - \kappa = -\kappa^2 < 0$ . Hence  $f_{PGW}$  is log-concave for all  $\kappa > 0$ .

**Appendix E: proof that  $\log h_{F:GF}$  is log-concave when  $\kappa > 1$  and is not log-concave when  $\kappa < 1$**

When  $\kappa > 1$ , the proof of log-concavity of  $\log h_{F:GF}$  follows similar lines to that in Appendix A. Define

$$G_\kappa(x) = \int_{e^x}^{\infty} \frac{y^{\kappa-1}}{(1+y)^{\kappa+\tau}} dy$$

(noting that  $G_\kappa$  also depends on  $\tau$ ), so that

$$\log h_{F:GF}(x) = \kappa x - (\kappa + \tau) \log(1 + e^x) - \log G_\kappa(x),$$

and define  $L(x) = \exp\{(\kappa - 1)x\}/(1 + e^x)^{\kappa+\tau}$ . In this case,

$$G'_\kappa(x) = -e^x L(x) \quad \text{and} \quad G''_\kappa(x) = \left( \frac{(\kappa + \tau)e^x}{1 + e^x} - \kappa \right) e^x L(x)$$

and, by integration-by-parts,

$$(\kappa + \tau)G_{\kappa+1}(x) = e^x L(x) + \kappa G_\kappa(x)$$

and

$$(\kappa + \tau - 1)G_\kappa(x) = (1 + e^x)L(x) + (\kappa - 1)G_{\kappa-1}(x).$$

Now,

$$(\log h_{F:GF})''(x) = -\frac{(\kappa + \tau)e^x}{(1 + e^x)^2} - (\log G_\kappa)''(x) = \frac{e^x}{G_\kappa^2(x)} \mathcal{B}_\kappa(x)$$

where

$$\begin{aligned}
\mathcal{B}_\kappa(x) &= e^x L^2(x) + \left( \kappa - \frac{(\kappa + \tau)e^x}{1 + e^x} \right) L(x)G_\kappa(x) - \frac{(\kappa + \tau)}{(1 + e^x)^2} G_\kappa^2(x) \\
&= L(x)\{e^x L(x) + \kappa G_\kappa(x)\} - \frac{(\kappa + \tau)}{(1 + e^x)^2} G_\kappa(x)\{(1 + e^x)L(x) + G_\kappa(x)\} \\
&= \frac{(\kappa + \tau)}{(1 + e^x)} \{(\kappa + \tau - 1)G_\kappa(x) - (\kappa - 1)G_{\kappa-1}(x)\}G_{\kappa+1}(x) \\
&\quad - \frac{(\kappa + \tau)}{(1 + e^x)} G_\kappa(x)\{(\kappa + \tau)G_{\kappa+1}(x) - \kappa G_\kappa(x)\} - \frac{(\kappa + \tau)}{(1 + e^x)^2} G_\kappa^2(x) \\
&< \frac{(\kappa + \tau)}{(1 + e^x)} \left[ (\kappa + \tau - 1)G_\kappa(x)G_{\kappa+1}(x) - (\kappa - 1)G_\kappa^2(x) \right. \\
&\quad \left. - (\kappa + \tau)G_\kappa(x)G_{\kappa+1}(x) + \kappa G_\kappa^2(x) - \frac{1}{(1 + e^x)} G_\kappa^2(x) \right] \\
&= \frac{(\kappa + \tau)}{(1 + e^x)} G_\kappa(x) \left\{ \frac{e^x}{1 + e^x} G_\kappa(x) - G_{\kappa+1}(x) \right\} < 0.
\end{aligned}$$

The first inequality arises from the fact that  $\kappa > 1$  and the Cauchy-Schwartz inequality applied to the second term, that is,  $G_{\kappa-1}(x)G_{\kappa+1}(x) > G_\kappa^2(x)$ . The final inequality follows because

$$\begin{aligned}
G_{\kappa+1}(x) &= \int_{e^x}^{\infty} \frac{y^\kappa}{(1 + y)^{\kappa+\tau+1}} dy = \int_{e^x}^{\infty} \frac{y^{\kappa-1}}{(1 + y)^{\kappa+\tau}} \frac{y}{1 + y} dy \\
&> \frac{e^x}{1 + e^x} \int_{e^x}^{\infty} \frac{y^{\kappa-1}}{(1 + y)^{\kappa+\tau}} dy = \frac{e^x}{1 + e^x} G_\kappa(x),
\end{aligned}$$

$y/(1 + y)$  being an increasing function.

That  $\log h_{F;GF}$  can not be log-concave when  $\kappa < 1$  follows from consideration of the behaviour of  $(\log h_{F;GF})''(x)$  as  $x \rightarrow -\infty$ . Using formulae above and the facts that  $L(x) \sim \exp\{(\kappa - 1)x\}$  and  $G_\kappa(x) \sim 1$  as  $x \rightarrow -\infty$ , when  $\kappa < 1$ ,  $(\log h_{F;GF})''(x) \sim \kappa e^{\kappa x}$  which approaches zero from the positive side. (This contrasts with  $(\log h_{F;GF})''(x) \sim -(\kappa + \tau)e^x < 0$  when  $\kappa > 1$ .)