Orthant t copulas

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Summary

Copulas based on the t distribution are popular largely because of their asymptotic tail dependence which, in the bivariate case, is the same in both tails and depends, not very interpretably, on the joint action of two parameters. I introduce a simple device that moves all of the asymptotic tail dependence into one tail, its strength fully controlled by a single parameter. Some other properties of the resulting 'orthant t copulas' and indications of how widely the 'orthantising' notion might be applied are also included.

Some key words: Bivariate t distribution; Positive dependence; Symmetric multivariate distribution; Tail dependence.

1. Introduction

One popular way of constructing bivariate copulas is to employ the copulas associated with elliptically symmetric distributions, especially the bivariate normal or t distributions (Song, 2000, Fang et al., 2002, Frahm et al., 2003, Demarta & McNeil, 2005, Genest et al., 2007, Danaher & Smith, 2011). Normal distributions, of course, have relatively light tails, and this turns out to result in asymptotic tail independence; t distributions, on the other hand, have heavy tails and consequently asymptotic tail dependence (Embrechts et al., 2002, Frahm et al., 2003). This extremal dependence is responsible for much of the particular interest in t copulas, which seem to be especially popular in economics and finance.

I see two difficulties with the use of t copulas per se. The first is known (Balkema et al., 2010). Figure 1(a) displays the Cauchy copula density when the correlation coefficient $\rho = 0.5$; this is the t copula density when the degrees of freedom, ν, is set equal to 1. Figure 1(b) shows the distribution based on the Cauchy copula density with $\rho = 0.5$ when the marginals have been transformed to standard normal distributions. The strong dependence in the copula interacts with the light tails of the marginals to produce peculiarly shaped densities which would seem rarely if ever to be good models for a data cloud.

(a) (b)

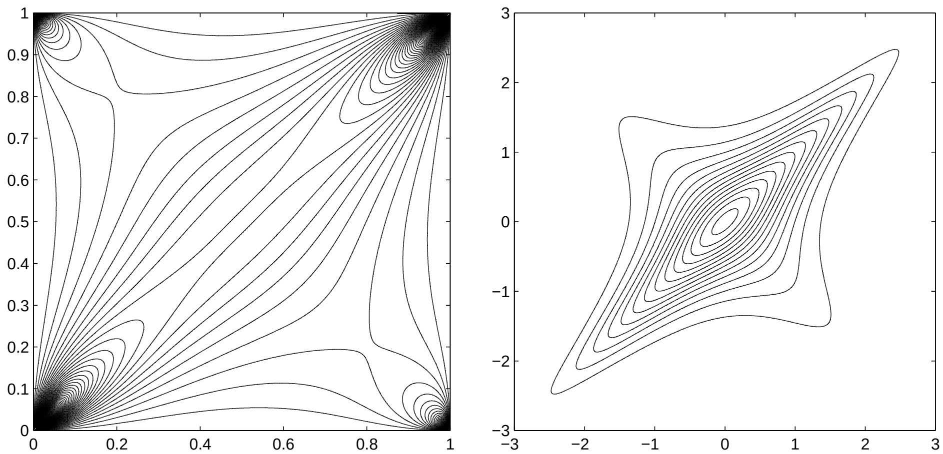

Figure 1: *(a) The Cauchy copula density with* $\rho = 0.5$*; (b) the density with Cauchy copula* ($\rho = 0.5$) *and normal marginals.*

Secondly, in general, the asymptotic tail dependence of a t copula depends on the values of both its parameters, ν and ρ (Embrechts et al., 2002, Frahm et al., 2003). Their interpretation, in the bivariate t context, of seemingly measuring tail weight and dependence separately is lost and their effects somehow funnelled into joint, and not readily interpretable, control of dependence.

In this paper, I put forward an extremely simple idea which avoids such odd shaped densities when marginals are light-tailed and allows the full range of asymptotic upper tail dependence under the control of a single dependence parameter. Note the mention of *upper* tail dependence, that is, dependence between coordinatewise maxima. If tail dependence is required in both upper and lower tails of a distribution, I have nothing to contribute in this paper. However, if, as I would suggest is usually the case especially since one is often dealing with positive random variables, a prime attribute to be captured is upper tail dependence, then this might be an attractive way to deploy t-type copulas.

The construction of this paper is also applicable more widely to copulas based on many symmetric underlying distributions, as will be seen. However, much emphasis remains on the t case because of a particular interest in distributions with, possibly strong, upper tail dependence. Sections 2 to 6 concentrate on the bivariate case, with multivariate extension delayed until Section 7.

2. 'Orthantising' symmetric copulas

The starting point is the copula of a doubly symmetric distribution, that is, a copula with density c satisfying

$$c(u,v) = c(1-u,v) = c(u,1-v) = c(1-u,1-v), \quad u,v \in (0,1)\times(0,1).$$

These arise from transformation to uniform marginals of sign symmetric distributions (Serfling, 2006) which have densities on $\mathbb{R}^2$ satisfying

$$f(x,y) = f(-x,y) = f(x,-y) = f(-x,-y), \quad x,y \in \mathbb{R}^2.$$

Examples include those associated with spherically symmetric distributions in general, such as the t distributions with $\rho = 0$ in particular. The corresponding Cauchy copula is shown in Fig. 2(a).

Now define the orthant copula associated with c to have density

$$c_o(u,v) = c\left(\tfrac{1}{2}(u+1), \tfrac{1}{2}(v+1)\right), \quad u,v \in (0,1)\times(0,1). \tag{1}$$

This copula density can be thought of as being an 'orthantised', that is, truncated to the positive orthant and renormalised, version of c; equivalently, if $\{U,V\} \sim c$, where '$\sim$' denotes 'follows the distribution with density', then $\{|2U-1|, |2V-1|\} \sim c_o$. It is also the copula density associated with the joint distribution of $\{|X|,|Y|\}$ if $X,Y \sim f$ where f is sign symmetric. As such, it is the bivariate copula analogue of 'half symmetric' distributions on $\mathbb{R}$. This construction is a very special, but apparently unexplored, case of 'threshold' or 'tail' copulas (Juri & Wüthrich, 2003).

The orthant t_ν copula density, for example, is given by

$$c_\nu(u,v) = \frac{1}{2\pi} \frac{(T_\nu^{-1})'\left(\frac{1}{2}(u+1)\right)(T_\nu^{-1})'\left(\frac{1}{2}(v+1)\right)}{\left[1+\nu^{-1}\left\{(T_\nu^{-1})^2\left(\frac{1}{2}(u+1)\right)+(T_\nu^{-1})^2\left(\frac{1}{2}(v+1)\right)\right\}\right]^{(\nu/2)+1}} \tag{2}$$

where T_ν^{-1} is the quantile function of the t_ν distribution.

It is easy to show that if C is the copula distribution function associated with c, then C_o, the copula distribution function associated with c_o, is given by

$$C_o(u,v) = 4C\left(\tfrac{1}{2}(u+1), \tfrac{1}{2}(v+1)\right) - 1 - u - v, \quad u,v \in (0,1)\times(0,1). \tag{3}$$

The orthant Cauchy copula density is shown in Fig. 2(b). Removal of the peaks in the Cauchy copula density at $(0,1)$ and $(1,0)$ results in more realistic looking densities, witness the version of Fig. 2(b) transformed to standard normal marginals in Fig. 2(c), and compare with Fig. 1(b). Moreover, removal of the peak at $(0,0)$ has resulted in the upper tail dependence being strengthened, in fact, doubled, as shown in the next section.

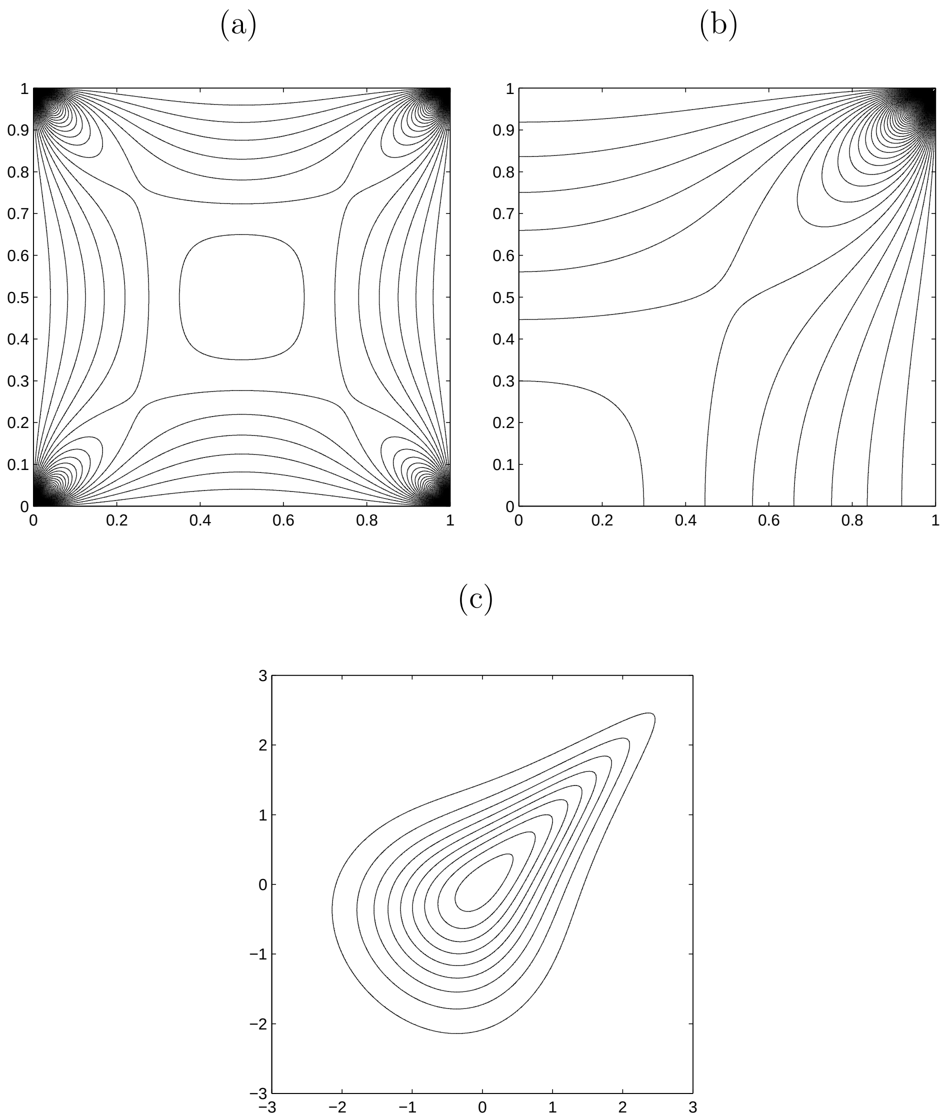

Figure 2: *(a) The Cauchy copula density with* $\rho = 0$*; (b) the orthant Cauchy copula density; (c) the density with orthant Cauchy copula and normal marginals.*

3. Tail dependence

The asymptotic upper tail dependence associated with copula C is

$$\lambda_U = \lim_{u\to 1} \frac{1-2u+C(u,u)}{1-u}$$

(Joe, 1993, 1997). This takes values in $[0,1]$, 0 corresponding to asymptotically independent upper tails, greater values to increasing amounts of asymptotic upper tail dependence. The properties of t copulas with respect to asymptotic dependence turn out to be shared with any copulas based on elliptical densities with regularly varying tails (Schmidt, 2002, Hult & Lindskog 2002). In particular, Hult & Lindskog's (2002) Fig. 2 indicates that λ_U is bounded above by 1/2 for such copulas when $\rho = 0$. Larger λ_U arises from $\rho \neq 0$ in addition to appropriate specification of ν. Properties of orthant t copulas in this section also apply, without further mention, to orthant copulas based on other elliptical densities with regularly varying tails.

Orthant t copulas do not suffer from any restriction on the value of λ_U, while depending only on the single parameter ν. In fact, using (3),

$$\begin{aligned}
\lambda_{U,o} &= \lim_{u\to 1} \frac{1-2u+C_o(u,u)}{1-u} \\
&= \lim_{u\to 1} \frac{4\{C\left(\frac{1}{2}(u+1), \frac{1}{2}(u+1)\right) - u\}}{1-u} \\
&= \lim_{u\to 1} \frac{4\{\lambda_U(1-u)/2 + u - u\}}{1-u} = 2\lambda_U.
\end{aligned}$$

Remarkably, it seems that orthantising the copula has had the effect of transferring all the tail dependence that was apportioned equally between lower and upper tails into the upper tail.

There are several ways of writing λ_U, and hence $\lambda_{U,o}$, for t copulas (Embrechts et al., 2002, Hult & Lindskog, 2002, Schmidt, 2002, Frahm et al., 2003). One such reduces when $\rho = 0$ to

$$\lambda_{U,o} = 4\,\bar{T}_{\nu+1}(\sqrt{\nu+1})$$

where $\bar{T}_\nu$ is the survival function of the t_ν distribution. This quantity is plotted as a function of ν in Fig. 3; see also Hult & Lindskog (2002, Fig. 2). It is confirmed mathematically in the Appendix that $\lambda_{U,o}$ is a decreasing function of $\nu > 0$ with $\lim_{\nu\to 0} \lambda_{U,o} = 1$ and $\lim_{\nu\to\infty} \lambda_{U,o} = 0$. Orthant t copulas therefore afford the full range of asymptotic upper tail dependence, the amount of that dependence depending solely and monotonically on the value of ν.

That *all* of the tail dependence in the t copula has gone into the upper tail dependence of the orthant t copula is confirmed by considering the latter's lower tail

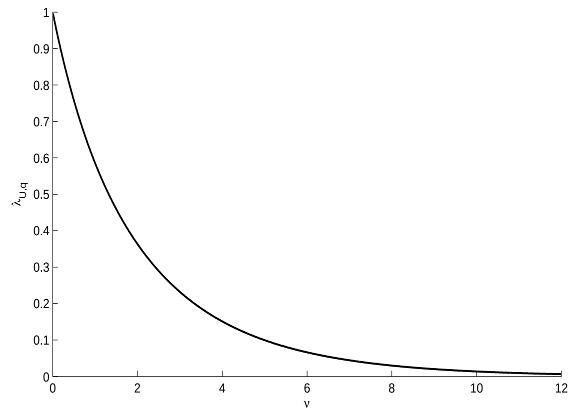

Figure 3: *The asymptotic upper tail dependence $\lambda_{U,o}$ of the orthant t copula plotted against ν.*

dependence. This is given (Joe, 1993, 1997) by

$$\begin{aligned}\lambda_{L,o} &= \lim_{u\to 0}\frac{C_o(u,u)}{u} = \lim_{u\to 0}\frac{4C\left(\frac{1}{2}(u+1),\frac{1}{2}(u+1)\right)-1-2u}{u}\\ &= \lim_{u\to 0}\frac{2u\{C^{10}\left(\frac{1}{2},\frac{1}{2}\right)+C^{01}\left(\frac{1}{2},\frac{1}{2}\right)-1\}}{u} = 0.\end{aligned}$$

Here, I have employed a Taylor series expansion and used the facts that, for a doubly symmetric copula, $C(\frac{1}{2},\frac{1}{2}) = \frac{1}{4}$, $C^{10}(\frac{1}{2},\frac{1}{2}) = dC(u,v)/du|_{\frac{1}{2},\frac{1}{2}} = \frac{1}{2}$ because $dC(u,v)/du = P(V \le v|U=u)$ where $(U,V)\sim c$, and similarly for $C^{01}(\frac{1}{2},\frac{1}{2})$. That is, the lower tails of the orthant t copula are asymptotically independent.

4. Positive dependence

Orthant t copulas are non-asymptotically positively dependent in the very strong sense that their densities are totally positive of order 2 (TP_2). This implies a number of further dependence properties which in turn ensure nonnegativity of scalar dependence measures such as Pearson correlation, Kendall's tau and Spearman's rho (Joe, 1997, Balakrishnan & Lai, 2009). An informative way to see that orthant t copula densities $c_\nu(u,v)$ are TP_2 is (Balakrishnan & Lai, 2009, p.170) to verify the nonnegativity at all points $u,v \in (0,1)\times(0,1)$ of the local dependence function $\gamma_\nu(u,v) = \partial^2 \log c_\nu(u,v)/\partial u\partial v$ (Holland & Wang, 1987). Since γ changes with marginal transformation in the same way as density functions do, nonnegativity of γ_ν follows from nonnegativity of the local dependence function of the orthantised symmetric t distribution with density

$$f_\nu(x,y) = \frac{2}{\pi}\frac{1}{\{1+\nu^{-1}(x^2+y^2)\}^{(\nu/2)+1}}, \qquad x,y>0;$$

this affords a very simple calculation of γ.

The local dependence function for the entire bivariate symmetric Cauchy distribution is shown in Fig. 2 of Jones (1996). There, as for all bivariate symmetric t distributions, "Random variables X and Y are positively associated in the first and third quadrants and negatively associated in the second and fourth" (Holland & Wang, 1987). The symmetry of the positive and negative dependences is typically averaged out by summary measures to produce zero overall dependence. See also Abdous et al. (2005) for further observations on the relatively less strong positive dependence of elliptical copulas in general. In this paper, by 'orthantising', we simply retain only the, possibly considerable, positive association in the first orthant.

By the way, negative association in a t-type copula could be introduced by retaining its second or fourth orthant, or equivalently by working with the distribution of U and $1-V$. However, this would destroy the positive dependence between maxima that is an important feature of the orthant t copula.

5. Some explicit formulae

Explicit formulae are not, in general, available for t copulas. They are, however, available for the two orthant t copulas associated with univariate t distributions with explicit expressions for their quantile functions. These have 1 and 2 degrees of freedom, respectively. The orthant Cauchy, t_1, copula — which can be obtained from (2) or via formulae in, for example, Balakrishnan & Lai (2009, Section 9.9) — has density

$$c_1(u,v) = \frac{\pi}{2}\frac{1}{\cos^2\left(u\frac{\pi}{2}\right)\cos^2\left(v\frac{\pi}{2}\right)\left\{1+\tan^2\left(u\frac{\pi}{2}\right)+\tan^2\left(v\frac{\pi}{2}\right)\right\}^{3/2}}$$

and distribution function

$$C_1(u,v) = \frac{2}{\pi}\tan^{-1}\left\{\frac{\tan\left(u\frac{\pi}{2}\right)\tan\left(v\frac{\pi}{2}\right)}{\sqrt{1+\tan^2\left(u\frac{\pi}{2}\right)+\tan^2\left(v\frac{\pi}{2}\right)}}\right\};$$

the orthant t_2 copula has density

$$c_2(u,v) = \frac{4}{\pi}\frac{\sqrt{(1-u^2)(1-v^2)}}{(1-u^2v^2)^2}$$

and distribution function

$$C_2(u,v) = \frac{2u}{\pi}\tan^{-1}\left(v\sqrt{\frac{1-u^2}{1-v^2}}\right) + \frac{v}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left\{\frac{1+u^2v^2-2u^2}{2u\sqrt{(1-u^2)(1-v^2)}}\right\}\right].$$

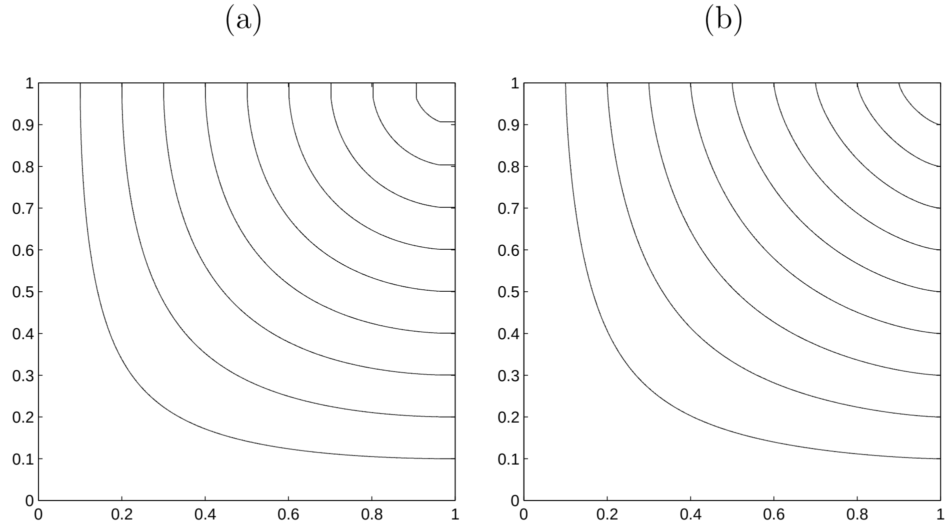

Figure 4: *Orthant t copula distribution functions when (a) $\nu = 1$, (b) $\nu = 2$. From the left, contours are at 0.1(0.1)0.9.*

The veracity of the formulae for C_1 and C_2 is best checked by confirming that $C_i(u,1) = u$, $C_i(1,v) = v$ and $\partial^2 C_i(u,v)/\partial u \partial v = c_i(u,v)$, $i = 1,2$; these copula distribution functions are shown in Fig. 4, and c_1 in Fig. 3(b). The asymptotic upper tail dependence functions for these two copulas can be explicitly checked to be $2 - \sqrt{2} \simeq 0.586$ and $1 - (2/\pi) \simeq 0.363$ respectively, in accordance with Fig. 3.

In addition, of course, as $\nu \to \infty$ the orthant Gaussian copula is obtained, which is the independence copula $C(u,v) = uv$.

6. Likelihood inference

Likelihood inference for parameters of distributions based on the orthant t copula is very similar to likelihood inference for parameters of distributions based on the t copula and existing software such as the `fitCopula` routine of the `copula` package in `R` (Yan, 2007) can be used. Adaptations are that, from (1), the uniformised data $\{U_i, V_i\}$, $i = 1, ..., n$, be transformed to $\{\frac{1}{2}(U_i + 1), \frac{1}{2}(V_i + 1)\}$, $i = 1, ..., n$, and, in a simplification, the correlation parameter ρ be set to zero, and not estimated. This equivalence is because the fitting of the t copula with $\rho = 0$ — or any other doubly symmetric copula — effectively works only with the values of $\{|2U_i - 1|, |2V_i - 1|\}$, $i = 1, ..., n$, anyway.

The theoretical work of Dakovic & Czado (2011) applies, in simplified form, for $\rho = 0$, to orthant t copulas also. Note that since the parameter being estimated is the tail dependence parameter, there is no need to give it special treatment as can happen when a correlation parameter is also present (Klüppelberg et al., 2008).

7. Multivariate extension

This paper has entirely concerned bivariate copulas so far. However, t copulas, and more generally copulas based on spherically symmetric and other sign symmetric distributions, are amongst the classes of copulas that have natural multivariate extensions, and this translates to orthant t copulas too. In d dimensions, sign symmetry means that $\{q_1X_1, ..., q_dX_d\}$ has the same distribution as $\{X_1, ..., X_d\}$ for all $q_i \in \{-1, 1\}$, $i = 1, ..., d$. Accordingly, the random variables $U_1, ..., U_d$ following the associated copulas are such that $\{q_1(2U_1 - 1), ..., q_d(2U_d - 1)\}$ has the same distribution as $\{2U_1 - 1, ..., 2U_d - 1\}$. In obvious extension of the bivariate case, the orthant copula associated with such copulas is the distribution of $\{|2U_1 - 1|, ..., |2U_d - 1|\}$, and is the copula associated with the distribution of $\{|X_1|, ..., |X_d|\}$.

The survival function of the d-variate orthant copula, $\bar{C}_o(u_1, ..., u_d)$ can then be written in terms of the original d-dimensional sign-symmetric copula survival function $\bar{C}(u_1, ..., u_d)$ as

$$\begin{aligned}
\bar{C}_o(u_1, ..., u_d) &= P\left(|2U_1 - 1| > u_1, ..., |2U_d - 1| > u_d\right) \\
&= P\left(2U_1 - 1 > u_1, ..., 2U_d - 1 > u_d | U_1 > \tfrac{1}{2}, ..., U_d > \tfrac{1}{2}\right) \\
&= 2^d P\left(U_1 > \tfrac{1}{2}(u_1 + 1), ..., U_d > \tfrac{1}{2}(u_d + 1)\right) \\
&= 2^d \bar{C}\left(\tfrac{1}{2}(u_1 + 1), ..., \tfrac{1}{2}(u_d + 1)\right).
\end{aligned}$$

From this, densities are related through

$$c_o\left(u_1, ..., u_d\right) = c\left(\tfrac{1}{2}(u_1 + 1), ..., \tfrac{1}{2}(u_d + 1)\right)$$

and, in the bivariate case, relationship (3) between copula distribution functions also follows immediately.

The bivariate advantage of removing any correlation parameter ρ in, for example, the t case, now translates into the multivariate disadvantage that the orthant t copula still only has a single parameter controlling dependence. Likelihood inference is, as in Section 6, like that for the original copula but simpler. It would seem, however, that a better way forward for multivariate quadrant t copulas would be to combine bivariate quadrant t copulas using one of the notions of pair copulas (Kurowicka & Cooke, 2006, Aas et al., 2009, Czado, 2010).

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Appendix

The asymptotic upper tail dependence of the orthant t copula can be written as

$$\lambda_{U,o} = 4\,\bar{T}_{\nu+1}(\sqrt{\nu+1}) = \frac{4\int_{\sqrt{\nu+1}}^{\infty}(1+\frac{t^2}{\nu+1})^{-(\nu+2)/2}dt}{2\int_0^{\infty}(1+\frac{t^2}{\nu+1})^{-(\nu+2)/2}dt} = \frac{2\int_1^{\infty}(1+x^2)^{-(\nu+2)/2}dx}{\int_0^{\infty}(1+x^2)^{-(\nu+2)/2}dx}.$$

The limiting values of $\lambda_{U,o}$ as $\nu \to \infty$ and as $\nu \to 0$ are readily obtained from the first and second of these ratio-of-integral representations, respectively. To see that $\lambda_{U,o}$ is decreasing, write $D = \int_0^{\infty}(1+x^2)^{-(\nu+2)/2}dx$ so that

$$\begin{aligned}
\frac{d\lambda_{U,o}}{d\nu} &= \frac{1}{D^2}\left\{\int_1^{\infty}\frac{1}{(1+x^2)^{(\nu+2)/2}}dx\int_0^{\infty}\frac{\log(1+x^2)}{(1+x^2)^{(\nu+2)/2}}dx\right.\\
&\qquad \left.-\int_0^{\infty}\frac{1}{(1+x^2)^{(\nu+2)/2}}dx\int_1^{\infty}\frac{\log(1+x^2)}{(1+x^2)^{(\nu+2)/2}}dx\right\}\\
&= \frac{1}{D^2}\left\{\int_1^{\infty}\frac{1}{(1+x^2)^{(\nu+2)/2}}dx\int_0^{1}\frac{\log(1+x^2)}{(1+x^2)^{(\nu+2)/2}}dx\right.\\
&\qquad \left.-\int_0^{1}\frac{1}{(1+x^2)^{(\nu+2)/2}}dx\int_1^{\infty}\frac{\log(1+x^2)}{(1+x^2)^{(\nu+2)/2}}dx\right\}\\
&< \frac{\log 2}{D^2}\left\{\int_1^{\infty}\frac{1}{(1+x^2)^{(\nu+2)/2}}dx\int_0^{1}\frac{1}{(1+x^2)^{(\nu+2)/2}}dx\right.\\
&\qquad \left.-\int_0^{1}\frac{1}{(1+x^2)^{(\nu+2)/2}}dx\int_1^{\infty}\frac{1}{(1+x^2)^{(\nu+2)/2}}dx\right\} = 0.
\end{aligned}$$

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