Distributions Generated by Transformation of Scale Using an Extended Schlömilch Transformation

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Abstract

Baker (2008) shows how more flexible densities on $\mathbb{R}^+$ can be generated from others by applying the Schlömilch transformation to the abscissa. Such "transformation of scale" is not even guaranteed to provide integrable functions in general. The appeal of the Schlömilch transformation is that it automatically does so; moreover, the normalising constant is unaffected and hence immediately available. In this paper, we fit the original Schlömilch transformation into a broader framework of novel extended Schlömilch transformations based on self-inverse functions, and propose the corresponding newly generated densities which also retain the same normalising constant. As well as providing parallels with, and extensions of, the many properties of the new densities developed by Baker, we investigate the skewness properties of both original and extended Schlömilch-based distributions via application of a recently proposed density-based approach to quantifying asymmetry.

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1 Introduction

One of the most popular approaches to generating new probability distributions from old is to employ *transformation of the random variable* concerned. If Z is a random variable following a distribution with probability density function g on support S_Z, say, and $t^{-1} : S_Z \to S_X$, say, is a differentiable monotone function, then the distribution of $X = t^{-1}(Z)$ has density function

$$f(x) = t'(x)g(t(x)), \tag{1.1}$$

$x \in S_X$. Logarithmic and power transformations (often in the guise of the Box-Cox transformation) are particularly famous examples of this approach.

A much less well studied alternative approach is to apply a transformation to the abscissa only; this is what we will refer to as *transformation of scale*. That is, consider densities of the form

$$f(x) \propto g(t(x)), \tag{1.2}$$

$x \in S_X$. Note immediately that $g(t(x))$ is not necessarily a density function itself: it might not be integrable. Even if it is, in general a new normalising constant is introduced and its explicit calculation may be far from easy. Existing examples of this approach are largely implicit; for example, one could think of generating exponential power distributions from the normal distribution in this way by replacing z in the latter by $|x|^{p/2}$, $p > 0$.

In recent insightful work, Baker (2008) proposes a family of transformation of scale distributions with the appealing, and remarkable, property that

$$f_S(x) = g(t_S(x)), \tag{1.3}$$

$x \in \mathbb{R}^+$. Note that f_S is a bona fide density function and its normalising constant is that of g (with no further calculation required). The particular transformation involved is the *Schlömilch transformation*,

$$t_S(x) = |x - b/x| \tag{1.4}$$

for any $b > 0$. This follows from an integral relationship which is the source of the Schlömilch transformation: for any function $\ell : \mathbb{R}^+ \to \mathbb{R}$,

$$\int_0^\infty \ell\left[\{x - (b/x)\}^2\right] dx = \int_0^\infty \ell(x^2) dx. \tag{1.5}$$

See Boros and Moll (2004, Chapter 13) for applications to integration and Baker (2008) for the historical genesis of the transformation as well as applications to statistics. Setting $\ell(x) = g(\sqrt{x})$ is, of course, the way to get from (1.5) to (1.3). (The slightly more general result in Theorem 13.2.1 of Boros and Moll, 2004, and Theorem 1 of Baker, 2008, corresponds precisely to introducing a scale parameter into (1.3) via $cf_S(cx) = cg(t_S(cx)) = cg(|cx - b/(cx)|)$, $c > 0$. We will study the canonical, $c = 1$, form of f_S and its extension in most of this paper; $c \neq 1$ should, of course, be reintroduced in practical work using these distributions.)

Here, we propose a more general "*extended Schlömilch transformation*" which is of the form

$$t_{ES}(x) = |x - s(x)|, \tag{1.6}$$

and which has the same integration and, hence, transformation of scale properties as the original Schlömilch transformation. Here, $s : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone decreasing function which is onto (informally, $s(0) = \infty$ and $s(\infty) = 0$) with the property of being *self-inverse* i.e.

$$s(s(x)) = x \quad \text{or} \quad s^{-1}(x) = s(x). \tag{1.7}$$

The associated mathematical integration theorem is proved in Section 2.1. In Section 2.2, we utilise a discussion of self-inverse functions given by Kucerovsky, Marchand and Small (2005) to come up with a number of alternatives to $s(x) = s_S(x) = b/x$, $b > 0$, of which a particular favourite will prove to be

$$s_1(x) = x - \frac{1}{\alpha}\log(e^{\alpha x} - 1), \tag{1.8}$$

$\alpha > 0$.

The remainder of the paper proceeds as follows. In Section 3, we extend the many results of Baker (2008) on properties of f_S to the extended form $f_{ES}(x) = g(t_{ES}(x))$. Some of these properties come out rather simply. Section 3.1 considers general g and Section 3.2 concentrates on the important case of monotone decreasing g and hence unimodal f_{ES}. In Section 4, we briefly indicate some special cases, particularly of my preferred version of f_{ES}, namely $f_1(x) = g(t_1(x))$ where $t_1(x) = |x - s_1(x)|$. Section 5 concerns skewness properties of f_{ES} as measured by a recent notion of density-based asymmetry (e.g. Critchley and Jones, 2008); it turns out that transformation of scale densities lend themselves especially well to this method of measuring asymmetry. This is investigated for f_S in Section 5.1 and for f_{ES} in Section 5.2. Finally, Section 6 comprises two short parts: Section 6.1 is on Schlömilch-based transformation of random variables and Section 6.2 on method of moment estimation.

2 Mathematical Aspects of the Extended Schlömilch Transformation

2.1. The extended Schlömilch transformation. Theorem 1 provides the extended Schlömilch transformation for evaluating integrals.

THEOREM 1. *Let $s : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone decreasing, onto, self-inverse transformation. Then,*

$$\mathcal{I} \equiv \int_0^\infty \ell\left[\{x - s(x)\}^2\right] dx = \int_0^\infty \ell(x^2) dx. \tag{2.1}$$

PROOF. Using the substitution $x = s(w)$,

$$\mathcal{I} = -\int_0^\infty \ell\left[\{w - s(w)\}^2\right] s'(w) dw. \tag{2.2}$$

Adding this to the original integral gives

$$2\mathcal{I} = \int_0^\infty \ell\left[\{w - s(w)\}^2\right] \{1 - s'(w)\} dw$$

so that, by using the transformation $v = w - s(w)$, we get

$$2\mathcal{I} = \int_{-\infty}^\infty \ell(v^2) dv = 2\int_0^\infty \ell(v^2) dv. \qquad \text{QED}$$

The original Schlömilch integration formula (1.5) corresponds to (2.1) with $s(x) = s_S(x) = b/x$.

2.2. Self-inverse functions on $\mathbb{R}^+$. The most immediate and apparent self-inverse function on $\mathbb{R}^+$ is the (scaled) reciprocal, s_S. The more general question of self-inverse functions on $\mathbb{R}$ was briefly considered by Kucerovsky et al. (2005, Section 2). They proposed two nice ways of generating such functions (which can readily be adapted, as follows, to decreasing, onto, functions on $\mathbb{R}^+$), the first being based on a pre-existing self-inverse function s:

(I) $s_H(x) = H^{-1}(s(H(x)))$ where $H : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing, onto, function;

(II) $s_h(x) = h(x)I(0 < x \le r) + h^{-1}(x)I(x > r)$, where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone decreasing, $h(0) = \infty$ and $h(r) = r$, for some $r > 0$.

Approach (I), in particular with $s(x) = 1/x$, has led me to the following examples. First, setting $H(x) = H_1(x) \equiv e^{\alpha x} - 1$, $\alpha > 0$, yields s_1 given by

(1.8); there is also a more general, two-parameter, version of this associated with $s_S(x) = b/x$, namely

$$s_{1,b}(x) = \frac{1}{\alpha} \log\left(1 + \frac{b}{(e^{\alpha x} - 1)}\right).$$

Second, setting $H(x) = H_1^{-1}(x)$ and reparametrising gives

$$s_2(x) = \exp\left\{\frac{\alpha}{\log(x+1)}\right\} - 1, \tag{2.3}$$

$\alpha > 0$. Third, $H(x) = \sinh \alpha x$ in (I) with s_S gives

$$s_{3,b}(x) = \frac{1}{\alpha} \log\left(\frac{b + \sqrt{b^2 + \sinh^2(\alpha x)}}{\sinh(\alpha x)}\right)$$

which reduces in the one-parameter case to

$$s_3(x) = \frac{1}{\alpha} \log\left(\frac{1 + \cosh(\alpha x)}{\sinh(\alpha x)}\right). \tag{2.4}$$

Fourth, the H^{-1} version of the above is

$$s_4(x) = \sinh\left(\frac{\alpha}{\sinh^{-1}(x)}\right). \tag{2.5}$$

Iterated and combined versions of these formulae are also available as are functions of the form $\{s_i(x^\gamma)\}^{1/\gamma}$, $i = 1, ..., 4$, $\gamma > 0$. This uses $H(x) = x^\gamma$ which has not been previously mentioned because it provides nothing new when applied to the reciprocal function.

The piecewise nature of the formulation of s_h in Approach (II) is *perhaps* less appealing than construction of s_H via Approach (I). One way to define h in Approach (II) is as $1/H$ from Approach I i.e.

$$s_h(x) = \frac{1}{H(x)} I(0 < x \le r) + H^{-1}\left(\frac{1}{x}\right) I(x > r)$$

for any of the H functions mentioned above, although explicit specification of r is not always possible. Alternatively, one might modify any of (1.8), (2.3)–(2.5) by setting $h_i(x) = s_i(x)$, $i = 1, ..., 4$.

Self-inverse function $s_1(x)$ in (1.8) will, along with $s_S(x) = b/x$, be particularly favoured in Section 4 where explicit invertibility of $x - s(x)$ will become an issue.

3 Basic Statistical Aspects of Extended Schlömilch Transformation-of-Scale

In this section, we study distributions with density of the form

$$f_{ES}(x) = g(t_{ES}(x)) = g(|x - s(x)|), \qquad x \in \mathbb{R}^+. \tag{3.1}$$

We will first give some properties that obtain for any density g on $\mathbb{R}^+$ (Section 3.1) and then concentrate on my preferred particular versions of (3.1) which derive from monotone decreasing g. All results in this section are extensions of those given by Baker (2008) for the case $s(x) = b/x$. (This fact will not be repeatedly acknowledged in what follows.)

3.1. Properties of f_{ES} derived for general g.

3.1.1. Tail behaviour. First, $f_{ES}(0) = f_{ES}(\infty) = g(\infty) = 0$; the behaviour of f_{ES} at 0 is perhaps a bit limiting compared with distributions (e.g. gamma, Weibull) which allow $f(0)$ to be non-zero and even infinite. The way that f_{ES} approaches 0 is governed by g and s as follows:

$$f_{ES}(x) \sim g(s(x)) \text{ as } x \to 0;$$

$$f_{ES}(x) \sim g(x) \text{ as } x \to \infty.$$

Note, in particular, that f_{ES} inherits its right-hand tail behaviour directly from that of g.

3.1.2. Expectations. The result of Theorem 1 in Section 2.1 immediately yields the following relationships for expected values of certain functions of X when X follows the distribution with density f_{ES}. Defining ℓ in Theorem 1 to be of the form $\ell(y) = m(\sqrt{y})g(\sqrt{y})$ where m is an arbitrary function, we immediately have that

$$E_{f_{ES}}\{m(|X - s(X)|)\} = E_g(m(Y))$$

(provided the latter exists). In particular, when $m(y) = y^r$ we have the 'moment' relationship

$$E_{f_{ES}}\{|X - s(X)|^r\} = E_g(Y^r). \tag{3.2}$$

Moreover, from (2.2) in the proof of Theorem 1, when $\ell(y) = g(\sqrt{y})$ we have that

$$E_{f_{ES}}\{s'(X)\} = -1. \tag{3.3}$$

In the case that $s(x) = b/x$, Baker (2008) combined formulae (3.2) and (3.3) to good effect to provide some relationships between lower order positive and negative moments (see, e.g., her Theorem 4).

3.1.3. 'Symmetry'. A density on $\mathbb{R}^+$ for which $f(x) = f(b/x)$ is said by Mudholkar and Wang (2007) to be R-symmetric. Formula (1) with $s(x) = b/x$ therefore gives a way of generating R-symmetric distributions. One could say that (1) in general generates 'S-symmetric' distributions where we hereby define f such that $f(x) = f(s(x))$ to be an S-symmetric density. It is unclear how far it is worth pursuing this concept.

3.2. Properties of f_{ES} derived from monotone g.

3.2.1. Unimodality. As Baker (2008) notes in the case of the Schlömilch transformation, if we choose g to be a *decreasing* density on $\mathbb{R}^+$, then f_{ES} will be a *unimodal* density on $\mathbb{R}^+$. To see this in general, observe that, except at the single point x_0 such that $x_0 = s(x_0)$,

$$f'_{ES}(x) = g'(|x - s(x)|)(1 - s'(x))\text{sign}(x - x_0) \tag{3.4}$$

which, since $g'(x) < 0$ and $s'(x) < 0$, $x > 0$, is positive for $x < x_0$ and negative for $x > x_0$. It follows that f_{ES} is unimodal with mode at x_0. Notice that $f_{ES}(x_0) = g(0)$. Notice also that if $s = s_h$ is obtained through Approach (II) in Section 2.2, then r there can be identified with x_0.

3.2.2. Differentiability at the mode. Note that

$$\lim_{x \to x_0^{\pm}} f'_{ES}(x) = \pm g'(0)(1 - s'(x_0)).$$

However, $1 - s'(x_0) = 2$. To see this, observe that differentiating $s(s(x)) = x$ yields $s'(s(x))s'(x) = 1$ which, at $x = x_0$, yields $\{s'(x_0)\}^2 = 1$; for decreasing s, $s'(x_0)$ is therefore equal to -1. It follows that there is a discontinuity in derivative of f_{ES} at x_0 unless $g'(0) = 0$. The latter constraint therefore provides, in this author's view as in Baker's, a more appealing subset of unimodal densities of type f_{ES}.

3.2.3. Mean-mode relationship.

THEOREM 2. *If f_{ES} given by (3.1) is unimodal and s is a strictly convex function, then $\mu > x_0$ where μ and x_0 are the mean and the mode of f_{ES}, respectively.*

PROOF.

$$\begin{aligned}\mu - x_0 &= \int_0^{x_0} (x - x_0) f_{ES}(x) dx + \int_{x_0}^{\infty} (z - x_0) f_{ES}(z) dz \\ &= \int_0^{x_0} \left[x - x_0 - s'(x)\{s(x) - x_0\}\right] f_{ES}(x) dx \\ &= \int_0^{x_0} \psi(x, x_0) f_{ES}(x) dx, \text{ say,}\end{aligned}$$

the second line arising by substituting $z = s(w)$ in the second integral in the first line. Now, if s is strictly convex on $\mathbb{R}^+$ then, inter alia,

$$s(x) - x_0 = s(x) - s(x_0) > -s'(x_0)(x_0 - x) = x_0 - x$$

so that $\psi(x, x_0) > (x_0 - x)(-s'(x) - 1)$. But for all $x < x_0$, strict convexity also implies that $s'(x) < s'(x_0) = -1$ so that $\psi(x, x_0) > 0$ and hence $\mu - x_0 > 0$. QED

3.2.4. Median-mode relationship.

THEOREM 3. *If f_{ES} given by (3.1) is unimodal and s is a strictly convex function, then $m > x_0$ where m and x_0 are the median and the mode of f_{ES}, respectively.*

PROOF.

$$\begin{aligned}F_{ES}(x_0) &= \int_0^{x_0} g(s(x) - x) dx = -\int_{x_0}^{\infty} g(w - s(w)) s'(w) dw \\ &< \int_{x_0}^{\infty} g(w - s(w)) dw = 1 - F_{ES}(x_0),\end{aligned}$$

the inequality arising from the fact that $s'(x) > s'(x_0) = -1$ for all $x < x_0$. It follows that $F_{ES}(x_0) < 1/2$ and hence that $m > x_0$. QED

3.2.5. Hazard function. Since the distribution function F_{ES} associated with f_{ES} is not, in general, available in closed form, then neither is the hazard function, $h_{ES}(x) = f_{ES}(x)/(1 - F_{ES}(x))$, of this distribution. It is clear that $h_{ES}(0) = 0$ and that as $x \to \infty$, $h_{ES}(x) \sim h_g(x)$ i.e. f_{ES} ultimately shares the hazard function of g.

4 Examples

Baker (2008) briefly discusses distributions of the form $g(|x - b/x|)$, $b > 0$, for the following choices of (monotone) $g(x)$: exponential, $(1 + x)e^{-x}/2$, a complementary incomplete gamma function density, the half-normal, the half-t, the half-exponential power, the half-generalised t and the Pareto. A pre-existing distribution that arises from this is a version of the root reciprocal inverse Gaussian (RRIG) distribution, which derives from the half-normal. This has density

$$f_{RRIG}(x) = \sqrt{\frac{2}{\pi}} e^b \exp\left\{-\frac{1}{2}\left(x^2 + \frac{b^2}{x^2}\right)\right\};$$

if X has distribution with density f, $X = 1/\sqrt{Y}$ where Y follows the inverse Gaussian distribution with parameters $\mu = 1/b$ and $\lambda = 1$. The RRIG distribution is a key example of an R-symmetric distribution in Mudholkar and Wang (2007).

The simplicity of $x - s_1(x) = \log(e^{\alpha x} - 1)/\alpha$, from (1.8), makes its absolute value perhaps the most attractive alternative transformation-of-scale to consider. $s_1(x)$ is convex. Distributions with density $f_1(x) = g(|x - s_1(x)|)$ all have the following specific properties, amongst others:

(i) $f_1(x) \sim g(-\log(\alpha x)/\alpha))$ as $x \to 0$;

(ii) $E\{(\exp(\alpha x) - 1)^{-1}\} = 1$;

(iii) $x_0 = \log(2)/\alpha$.

By way of specific example, the distribution associated with exponential g is just the rather remarkable

$$f_{ESE}(x) = (e^{\alpha x} - 1)^{\text{sign}(\alpha x - \log 2)/\alpha},$$

though this density is not continuously differentiable at x_0. The half-normal-based distribution has density

$$f_{\text{ESHN}}(x) = \sqrt{\frac{2}{\pi}} \exp\left\{-\frac{1}{2\alpha^2}\log^2\left(e^{\alpha x} - 1\right)\right\}; \tag{4.1}$$

it is displayed for several values of α in Figure 1. Note that the half-normal distribution is recovered as $\alpha \to \infty$. For $\alpha \to 0$, the (standard) normal distribution itself is the limiting case. To see this, the appropriate normalisation to consider is $\lim_{\alpha\to 0} \sigma f_{\mathrm{ESHN}}(\sigma x + \mu)$ where $\sigma = 1/2$ and $\mu = \log(2)/\alpha$.

* * * Figure 1 about here * * *

5 Asymmetry

While the transformation of random variables approach is very closely related to the van Zwet (1964) notions of skewness and kurtosis orderings, the transformation of scale approach lends itself much more readily to density-based skewness and kurtosis measures as recently propounded in Critchley and Jones (2008). In the case of skewness, Critchley and Jones proposed investigation of an *asymmetry function* of the form

$$\gamma(p) = \frac{x_R(p) - 2x_0 + x_L(p)}{x_R(p) - x_L(p)}, \quad 0 < p < 1, \tag{5.1}$$

where $x_R(p)$ and $x_L(p)$ denote the right- and left-hand solutions to $f_{ES}(x) = pf_{ES}(x_0)$. (See also Avérous, Fougères and Meste, 1996, and Boshnakov, 2007, for essentially the same idea, but we stick with our notation here.) Such a function is especially informative for 'rooted' unimodal densities, which is just what f_{ES}'s given by (3.1) for monotone decreasing g are; here, rooted means that $f_{ES}(0) = f_{ES}(\infty) = 0$.

The asymmetry function depends on g through $c_g(p) \equiv g^{-1}(pg(0))$, $0 < p < 1$. Then, $x_L(p)$ and $x_R(p)$ satisfy

$$|x - s(x)| = c_g(p). \tag{5.2}$$

Much further progress can be made when (5.2) can be explicitly solved and this is the case for both $s_S(x) = b/x$ and $s_1(x)$ given by (1.8).

As explained in Section 5 of Critchley and Jones (2008), the scalar skewness measure $\gamma = 1 - 2F(x_0)$ of Arnold and Groeneveld (1995) arises by multiplying $\gamma(p)$ by $f(x_0)$ times its denominator and integrating over p. We also evaluate this measure for each special case below.

5.1. Asymmetry function associated with original Schlömilch transformation. Let us turn first to the Baker (2008) case with $f_S(x) = g(|x - (b/x)|)$. From (5.2) with $s(x) = b/x$ we find that

$$\{x_R(p), x_L(p)\} = \frac{1}{2}\left(\sqrt{c_g^2(p) + 4b} \pm c_g(p)\right).$$

It follows that

$$\gamma_S(p) = \frac{1}{c_g(p)}\left(\sqrt{c_g^2(p) + 4b} - \sqrt{4b}\right). \tag{5.3}$$

Notice that, whatever monotone g and value of $b > 0$ are chosen, $\gamma_S(p) > 0$; the Schlömilch transformation of scale always results in positively skew distributions (in this quite strong sense). Moreover, as a function of b, $\gamma_S(p)$ decreases. This, too, is a strong sense in which asymmetry decreases with increasing b. It is also the case that $\gamma_S(p)$ is a decreasing function of p (for fixed b) with $\lim_{p\to 0} \gamma_S(p) = 1$ and $\lim_{p\to 1} \gamma_S(p) = 0$ (as Critchley and Jones, 2008, show such asymmetry functions must be for distributions on $\mathbb{R}^+$). See Figure 2 for a demonstration of these properties in the case of g chosen to be the half-normal distribution, $c_g(p) = \sqrt{-2\log p}$ i.e. Figure 2 portrays the asymmetry function of the RRIG$(1/b, 1)$ distribution.

* * * Figure 2 about here * * *

Next, we apply the integration mentioned above to $\gamma_S(p)$ in order to obtain the Arnold and Groeneveld (1995) scalar skewness measure. It turns out that $\gamma_S = g(0)\{\iota_S - \sqrt{4b}\}$, say, where

$$\begin{aligned} \iota_S &= \int_0^1 \sqrt{c_g^2(p) + 4b}\, dp = -\{g(0)\}^{-1} \int_0^\infty \sqrt{x^2 + 4b} g'(x) dx \\ &= \sqrt{4b} - \{g(0)\}^{-1} E_g\left(Z/\sqrt{Z^2 + 4b}\right). \end{aligned}$$

It follows that

$$\gamma_S = E_g\left(Z/\sqrt{Z^2 + 4b}\right)$$

which is a decreasing function of b, dropping from 1 when $b = 0$ to 0 as $b \to \infty$. Of course, we have also newly calculated the value

$$F_S(x_0) = \frac{1}{2}\left\{1 - E_g\left(Z/\sqrt{Z^2 + 4b}\right)\right\}$$

which increases from 0 to 1/2 as b increases, providing a much less direct verification of Theorem 3 for this s function.

When g is the half-normal distribution, use of 3.362.2 and 8.250.1 of Gradshteyn and Ryzhik (1994) yields

$$\gamma_S = 2e^{2b}\{1 - \Phi(2\sqrt{b})\}$$

where Φ is the standard normal distribution function.

5.2. Asymmetry function associated with extended Schlömilch transformation based on (1.8). In the case that $f_1(x) = g(|\log(e^{\alpha x} - 1)|/\alpha)$, we obtain

$$\{x_R(p), x_L(p)\} = \frac{1}{\alpha}\log\{\exp(\pm\alpha c_g(p)) + 1\}.$$

It follows that

$$\gamma_1(p) = \frac{2}{\alpha c_g(p)}\log\left\{\cosh\left(\frac{\alpha c_g(p)}{2}\right)\right\}. \tag{5.4}$$

Again, $\gamma_1(p) > 0$ whatever is (monotone) g, $\alpha > 0$ or $0 < p < 1$. As a function of α (for fixed p), this asymmetry function increases, while as a function of p for fixed α, asymmetry decreases. These properties are illustrated in Figure 3 for density (4.1) by, again, taking g to be the half-normal density. The most striking thing about Figure 3, at a general level, is its similarity to Figure 2. At a more detailed level, some skewness functions in Figure 3 have slightly different shapes from those in Figure 2.

* * * Figure 3 about here * * *

A calculation of the same type as set out in Section 5.1 provides the Arnold-Groeneveld scalar skewness measure for s_1. It turns out to be

$$\gamma_1 = E_g\left(\frac{\sinh\alpha x}{1 + \cosh\alpha x}\right).$$

This is, of course, an increasing function of α, increasing from 0 when $\alpha = 0$ to 1 as $\alpha \to \infty$. The value of $F_1(x_0)$ is available as $(1 - \gamma_1)/2$.

By the way, in the course of proving some of the properties of $\gamma_1(p)$ above, a distribution function for a symmetric density on $\mathbb{R}$ arose which the author has not seen before:

$$F_s(z) = \frac{1}{2}\left\{1 + z^{-1}\log(\cosh(z))\right\}$$

with density

$$f_s(z) = \frac{1}{2z^2}\left\{z\tanh(z) - \log(\cosh(z))\right\}.$$

This distribution, which is smooth, is Cauchy-like in the sense that $f_s(|z|) \sim |z|^{-2}$ as $z \to \infty$.

6 Miscellanea

6.1. Extended Schlömilch-related transformation of random variables. Again let Z be a random variable following a distribution on $\mathbb{R}^+$ with density g and make the random variable transformation $Y = t(Z) = Z - s(Z)$. (Note that this transformation does not have the modulus sign of the extended Schlömilch transformation.) Then Y follows a distribution on $\mathbb{R}$ with density f given by (1.1). The versions of Y corresponding to $s_S(x)$ and $s_1(x)$ are, again, of particular interest because of the explicit invertibility of their t functions. In fact, the random variable transformation $t(Z) = Z - (1/Z)$ has been explored in Jones (2007), stressing the property that it behaves like Z for large values of Z and like $1/Z$ for small values of Z (and the correspondence between the weights of tails of g and of f that results). The random variable transformation $t(Z) = \log(e^{\alpha Z} - 1)/\alpha$ seems not to have been investigated. It, too, has interesting effects on tail weights. These arise from it also behaving like Z for large Z but like $\log(\alpha Z)/\alpha$ for small Z.

6.2. Initial simple parameter estimation. A thorough investigation of parameter estimation and statistical inference, e.g. via maximum likelihood, is deferred to another paper. Here, we note that, especially for the basic Schlömilch transformation, method of moments-type estimation is particularly straightforward. Let $X_1, ..., X_n$ be a random sample from a Schlömilch distribution with one parameter, e.g. b, involved in the transformation plus the scale parameter, c i.e. the distribution has density $cf_{ES}(cx; b)$. Then, (3.2) and (3.3) hold with X replaced by cX. Choose g to be scaled such that $E_g(Y^2) = V$ (often, $V = 1$, as when g is half a symmetric density on $\mathbb{R}$ with unit variance).

For clarity, focus on the original Schlömilch transformation with $s(x) = b/X$ (as in Baker, 2008, but Baker did not consider parameter estimation).

Define the sample quantities $s_r = n^{-1}\sum_{i=1}^n X_i^r$ for $r = -2, 2$. Then, the sample version of (3.3) reduces to

$$bc^2 s_{-2} = 1$$

while the $r = 2$ version of (3.2) yields

$$c^{-2}s_2 - 2b + b^2c^2s_{-2} = V.$$

These can be solved to yield simple and general "method of moments" estimates

$$\hat{b} = \frac{V}{s_2 s_{-2} - 1}, \qquad \hat{c}^2 = \frac{1}{V}\left(s_2 - \frac{1}{s_{-2}}\right).$$

Note that $s_2 > 1/s_{-2}$ by Jensen's inequality so that $\hat{b}, \hat{c}^2 > 0$ and the only dependence on the underlying g is through the value of V.

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Figure 1: Density function (4.1) for the extended Schlömilch distribution using s_1 with half-normal g and, from right to left, $\alpha = 0.2, 0.35, 0.5, 1, 2.5, 10$.

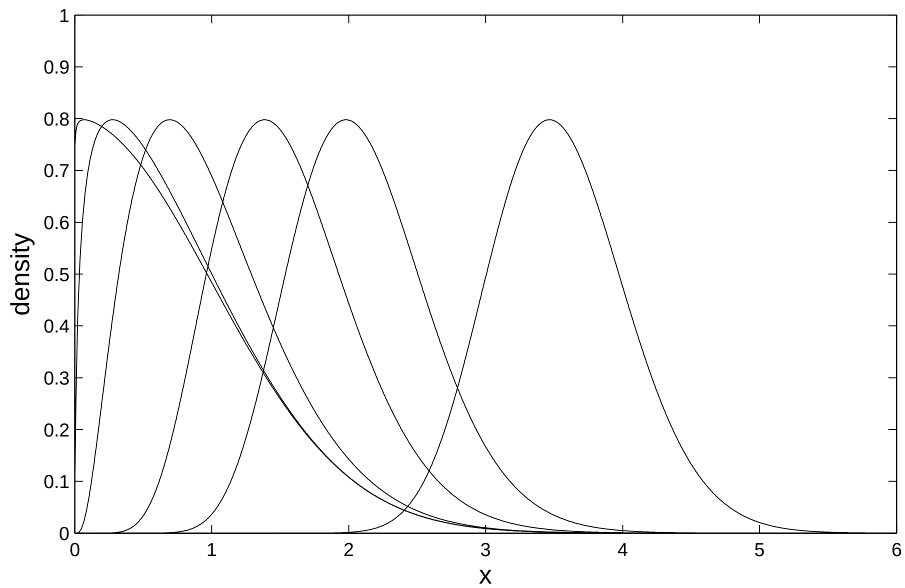

Figure 2: Skewness functions $\gamma_S(p)$ for the Schlömilch distribution with half-normal g and, in order of decreasing amount of skewness, $\sqrt{b} = 0.1, 0.5, 1, 2.5,$ $10, 100$.

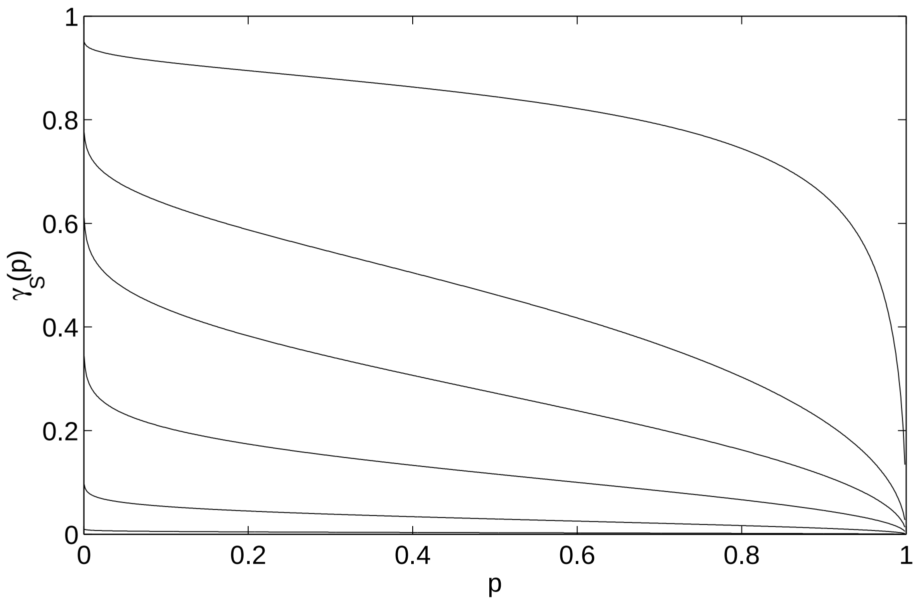

Figure 3: Skewness functions $\gamma_1(p)$ for the extended Schlömilch distribution using s_1 with half-normal g and, in order of increasing amount of skewness, $\alpha = 0.01, 0.1, 0.5, 1, 2.5, 10$.